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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 7, Number 2, September 1982

0273-0979/82/0000-0226/$02.25

The Hardy-Littlewood method, by R. C. Vaughan, Cambridge Tracts in Mathematics, vol. 80, Cambridge University Press, Cambridge, 1981, xii + 172 pp., $34.50.

A few years ago I heard a prominent algebraic number theorist exclaim: "What, the Hardy-Littlewood method is still alive? I thought it had been dead long ago". The book under review shows that the method is alive and well!

Let $\mathfrak{F}: \mathfrak{D} \to \mathbf{Z}$ be a map into the integers assuming each value at most finitely often. The number $N = N(\mathfrak{F}, \mathfrak{D})$ of zeros of $\mathfrak{F}$ is the constant term of the formal series

$$F(z) = \sum_{\mathbf{x} \in \mathfrak{D}} z^{\mathfrak{F}(\mathbf{x})}.$$

Assuming that F is analytic in the disk $|z| < 1$ with the possible exception of $z = 0$, we may invoke Cauchy's integral formula to obtain

$$N = \frac{1}{2\pi i} \int_C z^{-1} F(z)\, dz, \tag{1}$$

where C is a circle centered at 0 with radius $\rho < 1$. What is surprising is not this formula, but the way in which the integral on the right may often be evaluated or approximated so as to give information about diophantine problems.

Hardy and Ramanujan **[1918]** used this integral formula to obtain an asymptotic relation for the partition function, and to deal with the number of representations of integers by sums of squares. More generally, in a series of papers beginning in 1920, Hardy and Littlewood applied the formula to Waring's problem, i.e. the representation of integers n by sums of nonnegative kth powers:

$$n = x_1^k + \cdots + x_s^k. \tag{2}$$

Here $\mathfrak{D} = \mathbf{Z}^+ \times \cdots \times \mathbf{Z}^+$ where $\mathbf{Z}^+$ are the nonnegative integers, $\mathfrak{F}(\mathbf{x}) = \mathfrak{F}(x_1, \ldots, x_s) = x_1^k + \cdots + x_s^k - n$, and $N = N(k, s, n)$. Hardy and Ramanujan noted that in the case $k = 2$, i.e. in the case of squares, the integrand in (1) has a simple approximation when z is close to $e(a/q) = e^{2\pi i a/q}$ where a/q is rational with small denominator. If the radius ρ is sufficiently close to 1, then the circle of integration can be divided into a finite number of arcs, each of which is close to some such point $e(a/q)$, and hence a good estimate for the integral (1) may be given. The situation is more complicated when $k > 2$. The integrand still can be well approximated when z is *very* close to $e(a/q)$ with *very* small denominator, i.e. when z is on an arc very close to such a point $e(a/q)$. Hardy and Littlewood call these arcs "major arcs" and their complement on C the "minor arcs". They were fortunate to have available Weyl's [**1916**] paper to show that for large s the contribution of the minor arcs to the integral (1) is small. The Hardy-Littlewood method, also called "Circle Method", thus consists of the formula (1), together with a judicious division of the path of integration into major and minor arcs.

Vinogradov [**1928**] noted that when $\mathfrak{D}$ is finite, then (1) is valid with C the unit circle. This applies in particular to Waring's problem, for we may take $\mathfrak{D}$ to be $I(n) \times \cdots \times I(n)$ where $I(n)$ consists of the integers in $0 \leqslant x \leqslant [n^{1/k}]$, with [ ] denoting integer parts. Making the substitution $z = e(\alpha)$, one gets

$$N = \int_T g(\alpha)\, d\alpha \tag{3}$$

where $T = \mathbf{R}/\mathbf{Z}$ and where

$$g(\alpha) = \sum_{\mathbf{x} \in \mathfrak{D}} e(\alpha \mathfrak{F}(\mathbf{x})). \tag{4}$$

In fact (3) is obvious from first principles. Nowadays (3) is used instead of (1), and the "arcs" become intervals on T.

Hardy and Littlewood in [**1919**] could show that when $s > k \cdot 2^{k-1}$, then

$$N(k, s, n) = \mathfrak{J}\mathfrak{S} n^{(s/k)-1} + O(n^{(s/k)-1-\delta}), \tag{5}$$

where $\delta = \delta(k, s) > 0$ and where the constant implicit in O may depend on k and s. Here $\mathfrak{J} = \mathfrak{J}(k, s)$, the "singular integral", is defined in terms of the real manifold given by (2), while $\mathfrak{S} = \mathfrak{S}(k, s, n)$, the "singular series", depends on arithmetical properties of our equation. Furthermore, $\mathfrak{J} > 0$, and $0 < c_1(k, s) < \mathfrak{S} < c_2(k, s)$. It follows that $G(k) \leqslant k \cdot 2^{k-1} + 1$, where $G(k)$ is the least value of s such that (2) is soluble for every sufficiently large n. Hilbert [**1909**] in his solution to Waring's problem had only shown that $G(k)$ was finite.

The formula (5) is typical. Whenever the Hardy-Littlewood method works for a diophantine problem depending on a parameter n, the number $N(n)$ of solutions satisfies

$$N(n) \sim \mathfrak{J}\mathfrak{S} n^{\beta}$$

where β is a suitable exponent. Here

$$\mathfrak{J} = \psi(\infty), \qquad \mathfrak{S} = \prod_{\text{primes } p} \psi(p),$$

where $\psi(\infty)$ is defined in terms of the embedding of the underlying variety into the real space defined over $\mathbf{R} = \mathbf{Q}_\infty$, say, whereas $\psi(p)$ for a prime p depends on the embedding into the p-adic space defined over $\mathbf{Q}_p$. One could call $\psi(p)$

"the density of p-adic solutions". Thus it may be said that the Circle Method gives a quantitative version of the "local to global principle", from the local fields $\mathbf{Q}_p$ to the global field $\mathbf{Q}$.

The method is easiest to apply to "additive problems", i.e. for polynomials $\mathfrak{F}(x_1,\ldots,x_s) = \mathfrak{F}_1(x_1) + \cdots + \mathfrak{F}_s(x_s)$ and sets $\mathfrak{D} = I_1 \times \cdots \times I_s$. Waring's problem is of this type. For additive problems the function $g(\alpha)$ of (4) becomes $g(\alpha) = f_1(\alpha) \cdots f_s(\alpha)$ with

$$f_j(\alpha) = \sum_{x \in I_j} e\big(\alpha \mathfrak{F}_j(x)\big).$$

Chapter 1 of the present book contains an introduction and an historical background. In Chapter 2 the validity of the asymptotic formula (5) is proved for $s > 2^k$, which nowadays, after Hua's **[1938]** work, is easy. The functions $f_1,\ldots,f_s$ are, except for a factor $e(-\alpha n)$ in one of them, equal to

$$f(\alpha) = \sum_{x=0}^{N} e(\alpha x^k)$$

with $N = [n^{1/k}]$. In estimating the contribution to the integral (3) from the minor arcs $\mathfrak{m}$, one needs 2^k variables to apply Hua's inequality

$$\int_T |f(\alpha)|^{2^k} d\alpha \ll N^{2^k - k + \varepsilon},$$

and the remaining variable to estimate $|f(\alpha)|$ on $\mathfrak{m}$ by "Weyl's inequality". Chapter 3 deals with Vinogradov's **[1937]** solution to the ternary Goldbach's problem, and for the binary problem contains a proof that almost every even integer is a sum of two primes. Chapter 4 gives a better approximation (than did Chapter 2) to the integrand on the major arcs. In particular it is shown that $\mathfrak{S}(n) = \mathfrak{S}(k, s, n) > c_1(k, s) > 0$ when $s \geqslant 4k$. Chapter 5 deals (a) with Vinogradov's "mean value theorem", which is a bound, on the one hand for the number of solutions of a certain system of equations, and on the other hand for the mean value of $|S(f)|^{2s} = |\sum_{x=1}^{N} e(f(x))|^{2s}$, the mean value being taken over certain polynomials f of degree k. (b) by a remarkable argument from the mean to the particular (but see also Mordell's lemma in §7.2, and the proof of Theorem 4.1) a better bound for $S(f)$ is given than Weyl's. (c) the asymptotic formula (5) is proved for $s > c_3 k^2 \log k$, and finally, (d) the bound $G(k) \leqslant k(\log k)(3 + o(1))$ is derived. Although Vinogradov's approach has been simplified by Karatsuba and by Bombieri, the impression is as dazzling as ever. For the bound on $G(k)$ one sets $\mathfrak{D} = I_1 \times \cdots \times I_s$ with distinct sets I_j. About $2k \log k$ variables are needed for an improved inequality of Hua type, $k \log k$ variables for an improved inequality of Weyl type (i.e. part (b)), and $4k$ variables to make sure that $\mathfrak{S} > 0$. Chapter 6 brings Davenport's work on $G(k)$ for small k, in particular the relation $G(4) = 16$. In Chapter 7, Vinogradov's $G(k) \leqslant k(\log k)$ $(2 + o(1))$ is derived, which is the best known to date. Chapter 8 contains the author's formidable recent result that almost every integer is the sum of a square, a cube, and a fifth power.

The remaining three chapters are no longer connected with Waring's problem. Chapter 9 contains Birch's [**1957**] theorem that a homogeneous polynomial equation of odd degree k has a nontrivial integer solution if the number of variables exceeds $c_4(k)$. Chapter 10 deals with work of Roth and of Sárközy which is most interesting since the Circle Method is applied not to a given sequence such as the kth powers, but to arbitrary sequences of positive density. Finally, Chapter 11 explains Davenport and Heilbronn's [**1946**] variation of the method, which deals with diophantine inequalities rather than equations.

The book thus contains rather more material than its size would suggest. Single additive equations are treated fairly thoroughly, but little is said about systems of equations (see e.g. Davenport and Lewis [**1969**]) or nonadditive equations. In particular, there is no account of Davenport's [**1963**] work on the solubility of homogeneous cubic equations in 16 variables. Recently (to appear) the reviewer has shown that a system of r cubic forms has a rational zero if the number of variables exceeds $c_5 r^5$. Extending the result of Birch, the reviewer [**1980**] has shown that a cubic form $\mathfrak{F}$ of odd degree k with real coefficients has a nontrivial integer solution of $|\mathfrak{F}(\mathbf{x})| < 1$ if the number of variables exceeds $c_6(k)$. It is certain that much future work will deal with nonadditive problems.

The author's style is concise and might pose some difficulties for novices in the area. Often it is necessary to get out paper and pencil to fill in details.

The reviewer and many of his generation first learned the Circle Method from Davenport's very readable [**1962**] lecture notes. Vinogradov's books [**1947, 1971**] on trigonometrical sums, although on the whole more specialized, also deal with the Circle Method. The book under review is more economical in its presentation and more thorough on Waring's problem. The books by Greenberg [**1969**] on forms in many variables and by Igusa [**1978**] on forms of higher degree do not contain the Hardy-Littlewood method, but deal with related questions. Greenberg's book is elementary and algebraic in outlook, while Igusa's contains modern analytic tools.

Various analytic methods have now been developed to deal with diophantine equations. Linnik [**1960**] used his "dispersion method" inspired by probability theory to show that every large number is the sum of two squares and a prime. Hooley [**1981**] has developed his own method, capable of dealing with certain up to now intractable additive problems. Inter alia he has given another proof of the square plus cube plus fifth power theorem. These approaches appear to be limited to special additive problems. On the other hand Igusa's [**1978**] approach is quite general, and in particular he has proved the rationality of Poincaré series. Igusa is optimistic that his methods will yield concrete results on rational forms, although so far the classical problems here have not been notably advanced by the modern tools. The reviewer believes that eventually the modern viewpoint will help, especially for nonadditive problems, but that much of the present elementary and combinatorial machinery will remain, albeit in disguise.

The author does not mention these other methods. He does not even mention concepts such as torus, group, finite field, or p-adic number. And indeed, these concepts would not have shortened any of his arguments. The