NEW DEFECT RELATIONS
FOR MEROMORPHIC FUNCTIONS ON $\mathbb{C}^n$

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For meromorphic functions $f$ and $\varphi$ on the complex line $\mathbb{C}^1$, one considers the counting function $N(r, \varphi, f) = \int_1^r n(t, \varphi, f)t^{-1}dt$, where $n(t, \varphi, f)$ denotes the number of solutions of the equation $f = \varphi$ (counting multiplicities) on the disk $\{ |z| \leq t \}$. If $T(r, \varphi) = o(T(r, f))$, one defines the defect $\delta(\varphi, f) = \liminf[1 - N(r, \varphi, f)/T(r, f)]$ and observes that $0 \leq \delta(\varphi, f) \leq 1$ as in the case where $\varphi =$constant. In 1929, R. Nevanlinna [4, p. 77] asked if the defect relation

$$(*) \quad \sum_{j=1}^q \delta(\varphi_j, f) \leq 2$$

is valid for distinct meromorphic functions $\varphi_j$ with $T(r, \varphi_j) = o(T(r, f))$. The case where the $\varphi_j$ are constant is Nevanlinna’s fundamental defect relation [4]. (If $q = 3$, then $(*)$ follows immediately from the Nevanlinna defect relation.)

In 1939, J. Dufresnoy [3] showed that $\sum \delta(\varphi_j, f) \leq d + 2$ if $f$ is transcendental and the $\varphi_j$ are distinct polynomials of degree $\leq d$. In 1964, C.-T. Chuang [2] gave a general Second Main Theorem which yields $(*)$ for the case where $f$ is holomorphic (or more generally when $\delta(\infty, f) = 1$) and which generalizes the defect relation of Dufresnoy [3]. However, this question of Nevanlinna remains unanswered today even for polynomial $\varphi_j$, despite Nevanlinna’s assertion [4, p. 77] that $(*)$ “follows easily” for this case. If $f$ is a meromorphic function on $\mathbb{C}^n$, then a special case of a theorem of W. Stoll [7] (see also Vitter [8]) yields $(*)$ for constant $\varphi_j$ as in the classical Nevanlinna theory. (In fact, the results of Chuang [2] generalize easily to $\mathbb{C}^n$.) In this note we announce a new defect relation of the form $(*)$ for meromorphic functions on $\mathbb{C}^n$, $n \geq 2$.

If $f$ and $\varphi$ are distinct meromorphic functions on $\mathbb{C}^n$, we let $D(\varphi, f)$ denote the divisor on $\mathbb{C}^n$ given by the solution (with multiplicities) to the equation $f = \varphi$. We write $N(r, \varphi, f) = N(r, D(\varphi, f))$, where $N(r, D)$ denotes the counting function for $D$ as given in [1 or 7]. We easily obtain the First Main Theorem,

$$N(r, \varphi, f) + m(r, \varphi, f) = T(r, f) + T(r, \varphi) + c,$$

where the proximity term $m(r, \varphi, f) \geq 0$. Our main result is the following

SECOND MAIN THEOREM. Let $f$, $\varphi_1, \ldots, \varphi_q$ be distinct meromorphic functions on $\mathbb{C}^n$ ($q \geq n - 1$) such that

(i) $\text{rank}(\varphi_1, \ldots, \varphi_q) = n - 1$.

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(ii) \( \text{rank}(f, \varphi_1, \ldots, \varphi_{n-1}) = n. \)

Then

\[
(q - 2)T(r, f) \leq \sum_{j=1}^{q} N(r, \varphi_j, f) - N(r, R(f, \varphi_1, \ldots, \varphi_{n-1})) \\
+ O\left( \sum_{j=1}^{q} T(r, \varphi_j) + \log r T(r, f) \right).
\]

Here "rank" means the maximal rank of the derivative matrix, and \( R \) stands for the ramification divisor (given by the Jacobian determinant). The symbol \( \| \) means that the inequality is valid for all \( r > 0 \) outside a set of finite Lebesgue measure. (If \( f \) is of finite order and the \( \varphi_j \) are rational, then the inequality of the theorem is valid for all \( r > 0 \).)

As in the classical theory, we let \( \overline{N}(r, \varphi, f) \) denote the counting function obtained by reducing all multiplicities to 1. The Second Main Theorem can be restated as follows:

**Corollary 1.** Let \( f, \varphi_1, \ldots, \varphi_q \) be distinct meromorphic functions such that

\[ \text{rank}(f, \varphi_1, \ldots, \varphi_q) = \text{rank}(\varphi_1, \ldots, \varphi_q) + 1. \]

Then

\[
(q - 2)T(r, f) \leq \sum_{j=1}^{q} \overline{N}(r, \varphi_j, f) + O\left( \sum_{j=1}^{q} T(r, \varphi_j) + \log r T(r, f) \right).
\]

For example, if \( \varphi_j = \varphi_j(z_1, \ldots, z_{n-1}) \) and \( \partial f/\partial z_n \not\equiv 0 \), then \( f, \varphi_1, \ldots, \varphi_q \) satisfy the hypothesis of Corollary 1. More generally, we can let \( \varphi_j = \varphi_j(g_1, \ldots, g_p) \) where \( p \leq n - 1 \) and the \( g_k \) are meromorphic functions on \( \mathbb{C}^n \) such that \( \text{rank}(f, g_1, \ldots, g_p) = p + 1. \)

If \( T(r, \varphi) = o(T(r, f)) \), then we define the defect \( \delta(\varphi, f) \) as in the one variable case above, and we similarly let \( \Theta(\varphi, f) = \liminf[1 - \overline{N}(r, \varphi, f)/T(r, f)] \). As in the classical case, we have \( 0 \leq \delta(\varphi, f) \leq \Theta(\varphi, f) \leq 1 \). We now state our general defect relation, which follows immediately from Corollary 1.

**Corollary 2 (Defect Relation).** Let \( f, \varphi_1, \ldots, \varphi_q \) be as in Corollary 1. If \( T(r, \varphi_j) = o(T(r, f)) \) for \( 1 \leq j \leq q \), then

\[ \sum \delta(\varphi_j, f) \leq \sum \Theta(\varphi_j, f) \leq 2. \]

The proof of our Second Main Theorem uses the methods of \([1 \text{ and } 5]\) and the essential estimate given in the lemma below. We let

\[ \omega = (\sqrt{-1}/2\pi) \partial \overline{\partial} \log(|w^0|^2 + |w^1|^2) \]
denote the Fubini-Study 2-form on \( \mathbb{C}P^1 \), and we let

\[ \rho(a, b) = \frac{|a^1 b^0 - a^0 b^1|}{(|a^0|^2 + |a^1|^2)^{1/2}(|b^0|^2 + |b^1|^2)^{1/2}} \]
denote the chordal distance on $\mathbb{CP}^1$. We consider the function
\[
\gamma = \rho^{-2}(4q - 2\log \rho)^{-2}
\]
($\gamma$ blows up along the diagonal). Let $f, \varphi_1, \ldots, \varphi_q$ be as in the Second Main Theorem, and assume $q \geq n + 2$. We regard $f, \varphi_j$ as meromorphic maps into $\mathbb{CP}^1$. Let
\[
S = \text{supp}\left[ \sum_{j=1}^{q} D(\varphi_j, f) + R(f, \varphi_1, \ldots, \varphi_{n-1}) \right].
\]
We define the volume form $\Psi$ on $\mathbb{C}^n - S$ (which is a variant of the volume form given by Carlson and Griffiths [1]) by
\[
\Psi = \left[ \prod_{j=1}^{q} \gamma(\varphi_j, f) \right] f^*\omega \wedge \varphi_1^*\omega \wedge \cdots \wedge \varphi_{n-1}^*\omega.
\]
Recall that the Ricci form $\text{Ric} \Psi$ is given by $\text{Ric} \Psi = (\sqrt{-1}/2\pi) \partial \bar{\partial} \log h$ where $\Psi = h(\sqrt{-1} d\bar{z}_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (\sqrt{-1} d\bar{z}_n \wedge d\bar{z}_n)$. We let
\[
\theta = \text{Ric} \Psi + 2 \sum_{i=1}^{n-1} \varphi_i^*\omega \quad \text{on } \mathbb{C}^n - S.
\]
\[\text{LEMMA. } \theta \text{ is positive and } \theta^n > \lambda^{2q-2} \Psi \text{ on } \mathbb{C}^n - S, \text{ where}
\]
\[\lambda = \min_{j \neq k} \rho(\varphi_j, \varphi_k).
\]
The positivity of $\theta$ is easy to verify without any condition on the rank of the $\varphi_j$. However the volume-form inequality of the lemma is not true in general if $\text{rank}(\varphi_1, \ldots, \varphi_q) = n$. Details will appear in [6].

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REFERENCES


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