SELF-DUAL CONNECTIONS
AND THE TOPOLOGY OF SMOOTH 4-MANIFOLDS

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1. Introduction, statement of result. To any compact oriented 4-manifold $X$ there is associated a quadratic form $Q$, defined on the cohomology group $H^2(X; \mathbb{Z})$ by $Q(\alpha) = (\alpha \cup \alpha)[X]$. Poincaré duality requires that it be a "unimodular" form—given by a symmetric matrix of determinant $\pm 1$ with respect to any base for the torsion free part of $H^2$. It is known from arithmetic that there are many such forms that are positive definite and not equivalent (over the integers) to the standard form [4, Chapter 5]. The problem of finding which forms are realised by simply-connected 4-manifolds was raised, for example, in [3]; a partial answer for smooth 4-manifolds is announced here in the form of

THEOREM. If $X$ is a smooth, compact, simply-connected oriented 4-manifold with the property that the associated form $Q$ is positive definite, then $Q$ is equivalent, over the integers, to the standard diagonal form.

As a particular application, the theorem shows that it is impossible to remove smoothly, by surgery, all three hyperbolic factors in a K3 surface (which has quadratic form $E_8 + E_8 + (1,1) + (1,1)$) since this would give a simply-connected smooth 4-manifold with definite form $E_8 + E_8$.

2. Method of proof. I give, in this note, an outline of the proof; a detailed account will appear soon. The idea of the proof is to exploit topological information that emerges from a study of "self-dual connections" or "instantons"; I take [1] as a general reference for background in this area, and for notation. Suppose throughout that $X$ is a 4-manifold satisfying the hypotheses of the theorem, and that we are given some Riemannian metric.

There is, up to isomorphism, a unique principal $SU(2)$ bundle $P$ over $X$ with characteristic class $c_2(P)[X] = -1$. One forms the space of all equivalence classes of connections on $P$ as the quotient of the affine space $\mathcal{A}$ of connections by the action of the "gauge group" $\mathcal{G}$ of automorphisms of $P$. A Hausdorff topology descends to $\mathcal{A}/\mathcal{G}$ and the dense open subset representing irreducible connections can be made into a Banach manifold. On the other hand a reducible connection corresponds to a reduction of $P$ to an $S^1$ bundle and in the neighbourhood of such a point the space $\mathcal{A}/\mathcal{G}$ has the structure of

$$(\text{Real Banach Space}) \times (\text{Complex Banach Space}/S^1).$$

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The $S^1$ action arises as the action of the stabiliser in $G$ of a reducible connection.

A simple calculation with characteristic classes shows that the number of topologically distinct reductions of $P$, and so the number of components of the singular subset of $\mathcal{A}/G$, is given in terms of the form $Q$ by

$$n(Q) = \frac{1}{2} \# \{ \alpha \in H^2(X; \mathbb{Z}) : Q(\alpha) = 1 \}.$$

3. Self-dual connections. A connection of $P$ is called "self-dual" if its curvature $\Omega$ satisfies the equation $*\Omega = \Omega$. This property is invariant under $G$ so we may define the moduli space $M \subset \mathcal{A}/G$ of equivalence classes of self-dual connections on $P$.

Atiyah, Hitchin, and Singer [1, §6] showed that, provided a certain cohomology group vanishes, the irreducible elements of $M$ form a smooth finite-dimensional submanifold of $\mathcal{A}/G$. Their formula for the dimension gives in the special case at hand

$$\dim M = 8 - (\frac{3}{2})(\chi(X) - \tau(X)) = 5.$$ 

In general we do not know that the vanishing condition will be satisfied, but this is not important for the topological application since one may show

**Lemma 1.** For suitable generic perturbations of the self-duality equations the perturbed moduli space $M^* \subset \mathcal{A}/G$ is a smooth 5-manifold at all points representing irreducible connections.

This is a straightforward extension of a standard general position argument to an infinite-dimensional setting; using the methods of Kuranishi [2], or of Smale [5].

It is not obvious that there are any self-dual connections on $P$ at all. Their existence follows from a recent theorem of C. H. Taubes [6, Theorem 1.2], once one observes that an equivalent form of the hypothesis that the quadratic form be positive is the statement that all harmonic 2-forms on $X$ are self-dual.

Then, using the techniques of [6] together with results of K. Uhlenbeck [7, 8], and guided by what is known for the case when $X = S^4$ [1], one has a good understanding of the "boundary" of $M$.

**Lemma 2.** There is an open subset $U \subset M$ which is a smooth 5-manifold diffeomorphic to $X \times (0,1)$, and $M \setminus U$ is compact.

**Notes.** (i) In proving this lemma I use the fact that $X$ is simply connected.

(ii) The perturbed moduli space $M^*$ of Lemma 1 has the same boundary properties as $M$.

At each reducible connection in $M^*$ the 5-manifold inherits a quotient singularity from the ambient space; around such a point, $M^*$ has the structure of $C^3/S^1$: a cone on $CP^2$. $M$, and so $M^*$, meets each component of the reducible connections in $\mathcal{A}/G$ exactly once, so there are $n(Q)$ singular points. Thus there is a smooth compact 5-manifold with boundary the disjoint union of $X$ and $n(Q)$ copies of $CP^2$.

Finally, an argument based on properties of the index for families of operators shows that this 5-manifold is orientable. (This was first proved by M. F. Atiyah.)
4. Proof of theorem. Note first that for any unimodular positive form $Q$, $\nu(Q) \leq \text{rank}(Q)$ with equality if and only if $Q$ is equivalent to the standard form.

Then apply the fact that signature is an invariant of oriented cobordism to the 5-manifold above. The signature of $X$ is $\text{rank}(Q)$ so there must be at least $\text{rank}(Q)$ copies of $\mathbb{CP}^2$. Hence $\nu(Q) = \text{rank}(Q)$ and $Q$ is equivalent to the standard diagonal form.

REFERENCES


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