DIAGONALIZING MATRICES OVER OPERATOR ALGEBRAS

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1. Introduction. Let $A_0$ be a $C^*$-algebra and $A$ be the algebra of $n \times n$ matrices with entries in $A_0$. If $A_0$ acting on a (complex) Hilbert space $H_0$ is a faithful representation of $A_0$, then $A$ acting as matrices on the $n$-fold direct sum $H$ of $H_0$ with itself is a faithful representation of $A$. As a subalgebra of $B(H)$, the algebra of all bounded operators on $H$, $A$ acquires an adjoint and norm structure relative to which it is a $C^*$-algebra. This structure can be described independently of the representations—in particular, the operator in $B(H)$ adjoint to $(a_{jk})$ is the element of $A$ whose matrix has $a_{jk}^*$ as its $j,k$ entry. If $A_0$ is the (algebra of) complex numbers $\mathbb{C}$, then $A$ is the algebra of $n \times n$ complex matrices and each normal element $a$ can be “diagonalized”—that is, there is a unitary element $u$ in $A$ such that $uau^{-1}$ has all its nonzero entries on the diagonal.

With $A_0$ a general $C^*$-algebra, can each normal element of $A$ be diagonalized? In §2, we give a construction (based on homotopy groups of spheres) to show that this (general) question has a negative answer. The main result is discussed in §3. If $A_0$ is a von Neumann algebra, diagonalization of normal operators is always possible. More generally,

**THEOREM.** If $R_0$ is a von Neumann algebra, $R$ is the algebra of $n \times n$ matrices over $R_0$, and $S$ is a commutative subset of $R$ with the property that $a^*$ is in $S$ if $a$ is in $S$, then there is a unitary element $u$ in $R$ such that $uau^{-1}$ has all its nonzero entries on the diagonal for each $a$ in $S$.

2. An example. Let $A_0$ be the algebra $C(S^4)$ of continuous complex-valued functions on the 4-sphere $S^4$ and let $A$ be the algebra of $2 \times 2$ matrices with entries in $A_0$. View $S^3$ as the unit sphere in two-dimensional Hilbert space $C^2$ and consider the standard action of $SU(2)$ (the group of $2 \times 2$ unitary matrices of determinant 1) on $C^2$. The mapping that takes $u$ in $SU(2)$ to the vector $u(1,0)$ is a homeomorphism of $SU(2)$ onto $S^3$. From [2], $\pi_4(S^3)$ is the additive group of integers modulo 2. Let $u_0$ be an essential mapping of $S^4$ into $SU(2)$ (that is, into $S^3$). The algebra $A$ can be viewed as continuous mappings of $S^4$ into $B(C^2)$. Thus $u_0$ is a unitary (hence normal) element of $A$. Suppose $u$ is a unitary element of $A$ that diagonalizes $u_0$. Then $u(p)u_0(p)u(p)^{-1}$ is a $2 \times 2$ diagonal matrix over $C$ for each $p$ in $S^4$. Let $\theta(p)$ be the complex conjugate of the determinant of $u(p)$, let $u_1(p)$ be $[\theta(p)\ 0\ 0\ 1]$, and let $v(p)$ be $u_1(p)u(p)$. 

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Then $v(p)$ is in $SU(2)$, $v(p)u_0(p)v(p)^{-1} = u(p)u_0(p)u(p)^{-1}$, and $v$ is a unitary element in $A$. Let $f$ and $g$ be two continuous mappings of $S^4$ into $SU(2)$ that take a "base point" $p_0$ in $S^4$ onto (the base point) $I$ in $SU(2)$. Let $fg$ denote the mapping that assigns to $p$ in $S^4$ the group product $(f(p)g(p))$ in $SU(2)$. Let $\{f\}$, $\{g\}$, and $\{fg\}$ be the corresponding elements (homotopy classes) in $\pi_4(SU(2)) (= \pi_4(S^3))$. From [1], $\{f\}\{g\} = \{fg\}$. Moreover $\pi_4(SU(2))$ is abelian. Thus

$$\{v_0v^{-1}\} = \{v\}\{u_0\}\{v^{-1}\} = \{u_0\}\{v\}\{v^{-1}\} = \{u_0\}\{vv^{-1}\} = \{u_0\} \neq 0.$$ 

But $v(p)u_0(p)v(p)^{-1}$ is diagonal and in $SU(2)$ and hence has the form $\begin{bmatrix} \lambda(p) & 0 \\ 0 & \lambda(p) \end{bmatrix}$, where $|\lambda(p)| = 1$. Thus $v_0v^{-1}$ maps $S^4$ into a subset of $SU(2)$ homeomorphic to $S^1$ and $\{v_0v^{-1}\} = 0$—a contradiction. Hence $u_0$ cannot be diagonalized.

3. Matrices over von Neumann algebras. Let $R_0$, $R$, and $S$ be as in the theorem of §1. Let $e_j$ be the element in $R$ whose only nonzero entry is the identity at the $j,j$ position. Then $e_1, \ldots, e_n$ are $n$ orthogonal equivalent projections in $R$ with sum the identity element of $R$. Suppose we can find $n$ orthogonal equivalent projections $f_1, \ldots, f_n$ in $R$ with sum the identity element such that each $f_j$ commutes with every element of $S$. From various results in the comparison theory of projections in von Neumann algebras, we can conclude that $e_j$ and $f_j$ are equivalent in $R$ for $j$ in $\{1, \ldots, n\}$. Let $v_j$ be a partial isometry in $R$ with initial projection $f_j$ and final projection $e_j$. Then $\sum_{j=1}^n v_j$ is a unitary element $u$ in $R$ such that $uf_ju^{-1} = e_j$ for $j$ in $\{1, \ldots, n\}$. Since $f_j$ commutes with each $a$ in $S$ (by assumption), $uau^{-1}$ commutes with $e_j (= uf_ju^{-1})$ for each $j$ in $\{1, \ldots, n\}$. Hence $uau^{-1}$ is diagonal for each $a$ in $S$.

The problem then is: Can we find $f_1, \ldots, f_n$ with the properties described? Does the "relative commutant" of $S$ in $R$ contain $n$ orthogonal equivalent projections with sum the identity? We have little control over this relative commutant. From Zorn's lemma, $S$ is contained in some maximal abelian (selfadjoint) subalgebra $A$ of $R$. Each such $A$ is contained in the relative commutant. But $S$ may itself be such an $A$, in which case, the relative commutant is a maximal abelian subalgebra of $R$. Thus we must be prepared to (and it suffices to) find $f_1, \ldots, f_n$ as described in an arbitrary maximal abelian subalgebra of $R$. In effect, we must develop a comparison theory of projections in a maximal abelian subalgebra of $R$ relative to $R$. The last of a series of results leading to such a theory is

**Theorem.** If $R$ is a von Neumann algebra and each type $I_k$ central summand of $R$ is such that $k$ is divisible by $n$, then each maximal abelian subalgebra of $R$ contains $n$ orthogonal equivalent projections with sum the identity element of $R$. In particular, this is true of the von Neumann algebra of $n \times n$ matrices over a von Neumann algebra.

The full account of these results deals with the case where $R_0$ is countably decomposable to avoid complicated but peripheral higher cardinality considerations.
4. Related questions. There are a number of other avenues of study indicated by the foregoing discussion and results. We mention a few. For which compact Hausdorff spaces $X$ is diagonalization of normal matrices over $C(X)$ possible in general? For $2 \times 2$ matrices? For $3 \times 3$ matrices? What is the relation between "$n$-diagonalizability" and "$m$-diagonalizability"? Certain types of normal elements may be diagonalizable in all circumstances—which are they? What "relative comparison theory" is possible for other von Neumann subalgebras of von Neumann algebras? For $C^*$-subalgebras of a von Neumann algebra?

BIBLIOGRAPHY


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