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DONALD E. SARASON

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Winning ways for your mathematical plays, by Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy, Academic Press, London, 1982, vol. 1, *Games in general*, vol. 2, *Games in particular*, xxxii + 861 pp., $22.50 P/B, $64.50 H/B.

Winning ways is a masterpiece. We should have been disappointed were it anything less. Fifteen years in the preparation, and representing the collaboration of three mathematicians of extraordinary talent, the result is the most compelling and comprehensive treatment of mathematical games to appear in this century.

First, an enumeration of some of the things which this book is *not*. It has an empty intersection with "the Theory of Games" in the sense of von Neumann and Morgenstern [**6**]. More generally, it avoids discussion of "games" in which randomizing elements (the roll of dice, the shuffling of cards, the spinning of discs, or other methods of selecting a "move" in a stochastic fashion) play any role. This leaves full information, "deterministic" games such as chess, checkers (draughts), and Go, in which two players move alternately. However, these three examples of games actually played by adult humans are far too complicated to be analyzed in *Winning ways*.

Winning ways is published in two volumes. The first volume (WWI) is subtitled *Games in general*, while the second (WWII) is subtitled *Games in particular*. Each volume in turn consists of two parts. The four parts are associated successively with Spades, Hearts, Clubs, and Diamonds, but this has no underlying significance, and is for identification purposes only.

"Spade-work" (the first eight chapters) develops the generalized theory for analyzing and evaluating Nim-like games. The published analysis of Nim itself [**1**] goes back to C. L. Bouton in 1902. In 1939, P. M. Grundy [**5**] published a method for the recursive evaluation of positions in a relatively large class of Nim-like games, and this evaluation function for such a game became known for some time as the "Grundy function" (cf. [4]). Then an earlier reference, published in 1935–36 in the German language, in a Japanese journal, by R. P. Sprague [7] came to light, and for a transition period (cf. [2]), the Grundy function was renamed the Grundy-Sprague function. In *Winning ways* the next step has occurred: the Sprague-Grundy function. Perhaps the final step will be to drop Grundy entirely.

In a Nim-like game, two players (Left and Right) take turns depleting an initial inventory, each having full information, and with the last player able to take something being the *winner*. (This is the *normal* rule. In the *misère* form of the game, the last player is the *loser*.) There may be complicated restrictions on how much of the inventory can be removed at a single turn, but the rules must apply *impartially* to the two players in order for the Sprague-Grundy theory to apply. *Winning ways* is equally concerned with *partizan* (i.e. asymmetric) games.

Central to *Winning ways* is its method of assigning a *value* to a game (or to a position in a game). It is this subject which Conway undertook to exposit in *On numbers and games* (ONAG) [2], which was started later, but finished earlier, than *Winning ways*, and is thus both its descendant and antecendent. The reader looking for mathematical rigor, austerity, and elegance may well prefer the style of ONAG to that of *Winning ways*. The latter presents the games in the forefront in a lively and entertaining manner, develops the number system needed to evaluate game positions as a direct consequence of the positions themselves, and relegates formal theorems and proofs to the "Extras" section at the end of each chapter. In ONAG, Conway consciously reversed the process, first developing the appropriate number system axiomatically (in Part 0), and then illustrating the relationship to games (in Part 1). The serious mathematical reader will skip none of the "Extras" in *Winning ways*, and will keep a copy of ONAG nearby for supplementary reference.

The first few chapters in *Winning ways* must be read in order for most of the rest to be intelligible. All eight chapters of Part I are carefully and entertainingly written, and are well worth reading. The games introduced (Hackenbush, Domineering, Toads-and-Frogs, Cutcake, etc.) will be previously unfamiliar to most readers, and serve primarily to introduce the concepts appropriate to game evaluation in a systematic fashion. However, all of these games are "playable", and may provide brief entertainment for the reader. In most cases, as with Nim and Tic-Tac-Toe, once the complete analysis is understood by both players, there is little further reason for actual play.

The second part ("Changes of Heart"), consisting of Chapters 9 to 13, discusses modifications and generalizations of the type of games already considered. A game may be *impartial* (either player can make any of the allowed moves, provided only that it is his turn, as in Nim) or *partizan* (where the players are distinguished by the color of their "pieces", or even by the set of pieces assigned to them). The objective may be *normal* (to make the last legal move) or *misère* (to force the opponent to make the last move). The resources and potential duration of the game may be *finite* or *infinite*, in various combinations and degrees. Games are classified as *tame*, *restive*, and *wild*. To the extended real number system developed in Part 0 of ONAG, additional objects must be adjoined in order to characterize the value of certain moves. For example, when playing a sum of several games, there may be a move in one of the component games which inevitably wins (loses) for the player making it, no matter what is going on in the other component games. Such a move is called *sunny* (*loony*), denoted by an appropriate astrological symbol, and it is obvious that such a value for that component does not "add" in a normal way to the values of the other components. (Another unpleasant possibility is that a move in one component may force the entire sum of games to be an inevitable and interminable draw. Such a component is called a *dud*.)

The reader more interested in playable games than in oddities of game analysis will probably skip to Part 3, called "Games in Clubs", and consisting of Chapters 14 to 22. Here certain "children's games" are shown to have unexpectedly rich complexity. Thus, all of Chapter 16 is devoted to the game of "Dots-and-Boxes", and its reasonable variants. Having mastered this chapter, the reader will have little difficulty winning at Dots-and-Boxes against the great majority of uninitiated opponents, but will still not know a complete winning strategy on all $m \times n$ arrays of dots. Chapters 19, 20, and 21 are about partizan board games of pursuit, where one player has a single movable piece (the "quarry") that tries to elude entrapment by the forces of the second player, who typically has several pieces, but with less mobility than the quarry. Pre-existing games of this type which are analyzed include "Fox and Geese" (four checker men as geese attempting to blockade a checker king as the fox, in a game where "jumping" is forbidden); Epstein's [**3**], "Quadraphage" (a chess king on a large board tries to avoid being encircled by the *quadraphage*, or *square-eater*, which devours any square of its choice on each successive turn); "Hare and Hounds", the "French Military Hunt", and other games differing only by the sizes of the board, which is a portion of the octagon-and-square tiling of the plane, where three hounds attempt to encircle the lone hare.

It is not until Chapter 22 that Tic-Tac-Toe (Noughts-and-Crosses) is treated. Several well-disguised equivalent games are described. The complete decision tree for the standard 3×3 game is given. Generalizations to r-in-a-row on a "board" n on a side in k dimensions are considered. Many of these games have a drawing strategy based on pairings of squares, and some of these are shown, ascribed to A. W. Hales and R. I. Jewett, to J. L. Selfridge, and to the reviewer. Go-Moku is mentioned but not analyzed. More distant relatives of Tic-Tac-Toe, such as Hex and Bridgit, are also considered in Chapter 22.

Much of Part 3 owes a significant debt (amply acknowledged) to Chapter 10 of Richard A. Epstein's book [**3**], which appeared around the time the *Winning ways* collaboration was initiated. In this reviewer's opinion, Epstein's book is a gem which never received the attention it deserved.

Part 4 of *Winning ways*, "Solitaire Diamonds", consists of three final chapters which are not about "games" at all. Chapter 23 is essentially an analysis of "Peg Solitaire". Chapter 24, one of the longest in the book, treats "puzzles", including everything from Rubik's Cube to Polyominoes and Soma, Instant Insanity (or "The Tantalizer"), wire-and-string puzzles, the Tower of Hanoi, Sam Loyd's Fifteen Puzzle, MacMahon Squares, and finally (and