Q VALUED FUNCTIONS MINIMIZING DIRICHLET'S INTEGRAL
AND THE REGULARITY OF AREA MINIMIZING
RECTIFIABLE CURRENTS UP TO CODIMENSION TWO

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We announce several results of an extensive study [A] of the size of singular sets in oriented \( m \) dimensional surfaces which are area minimizing in \( m + 1 \) dimensional Riemannian manifolds. Our principal result is that the Hausdorff dimension of such a singular set does not exceed \( m - 2 \). Examples show this is the best possible such general estimate when \( l \geq 2 \), i.e., when branching singularities are possible. The general existence of such surfaces of least area is well known in a variety of settings [F, 5.1.6].

In order to obtain estimates on branching of area minimizing surfaces we were led to use Taylor's expansion in terms of first derivatives at 0 to approximate the nonparametric area integrand by Dirichlet's integrand. Accordingly, we study branched coverings of regions in \( \mathbb{R}^m \) which are graphs of multiple valued functions minimizing the integral of Dirichlet's integrand. As a central estimate in our analysis of area minimizing surfaces we show that the Hausdorff dimension of the branch set of such a minimizing covering does not exceed \( m - 2 \).

To state several results in more detail we use the terminology of [F]. Suppose that \( A \) is a bounded open subset of \( \mathbb{R}^m \) with smooth boundary, and let \( k, l, m, n, Q \) be positive integers with \( k \geq 3, l \leq n, \) and \( m \geq 2 \).

**INTERIOR REGULARITY OF ORIENTED AREA MINIMIZING SURFACES.** Suppose \( N \) is an \( m + 1 \) dimensional submanifold of \( \mathbb{R}^{m+n} \) of class \( k + 2 \) and that \( T \) is an \( m \) dimensional rectifiable current in \( \mathbb{R}^{m+n} \) which is absolutely area minimizing with respect to \( N \). Then there is an open subset \( U \) of \( \mathbb{R}^{m+n} \) such that \( \text{spt} \, T \cap U \) is an \( m \) dimensional minimal submanifold of \( N \) of class \( k \) and the Hausdorff dimension of \( \text{spt} \, T \sim (U \cup \text{spt} \, \partial T) \) does not exceed \( m - 2 \).

For such area minimizing \( T \) we have additionally

**SINGULARITY STRATIFICATION BY TANGENT CONE TYPE.** Whenever \( p \in \text{spt} \, T \sim \text{spt} \, \partial T \) and \( S \) is an oriented tangent cone to \( T \) at \( p \) then

\[
P(S) = \mathbb{R}^{m+n} \cap \{x : \theta^m(||S||, x) = \theta^m(||S||, 0) = \theta^m(||T||, p)\}
\]

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is either the point \{0\} or a linear subspace of \(\mathbb{R}^{m+n}\) with \(m-1 \neq \dim P(S) \leq m\). Furthermore, for each \(j \in \{0, 1, \ldots, m - 2, m\}\), the Hausdorff dimension of \((spt T \sim spt \partial T) \cap \{p : j = \sup \{\dim P(S) : S \text{ is an oriented tangent cone to } T \text{ at } p\}\}\) does not exceed \(j\).

We denote by \(\mathcal{Q}\) the space of all 0 dimensional integral currents \(V\) in \(\mathbb{R}^n\) for which \(\mathcal{Q} = \mathcal{M}(V) = \langle V, 1 \rangle\) with metric given by setting
\[
dist([p(1)] + \cdots + [p(Q)], [q(1)] + \cdots + [q(Q)]) = \inf \left\{ \left( \sum_{i=1}^{Q} |p(i) - q(\sigma(i))|^2 \right)^{1/2} : \sigma \text{ is a permutation of } \{1, \ldots, Q\} \right\}
\]
everywhere \(p(1), \ldots, p(Q), q(1), \ldots, q(Q) \in \mathbb{R}^n\). For Lipschitz \(\mathcal{Q}\) valued functions we show a Lipschitz extension theorem analogous to Kirszbraun's theorem, an almost everywhere \(Q\) fold affine approximation theorem analogous to Rademacher's theorem, and also show that each Lipschitz function \(A \to \mathcal{Q}\) induces a natural chain mapping of degree 0 from the chain complex of real flat chains having supports in \(A\) into the chain complex of real flat chains in \(\mathbb{R}^n\). In terms of Dirichlet's integral naturally defined for appropriate functions \(A \to \mathcal{Q}\) we have the following central results.

**Existence and Regularity of Dirichlet Integral Minimizing \(\mathcal{Q}\) Valued Functions.** For each appropriate function \(g : \partial A \to \mathcal{Q}\) there exists a (strictly defined but not necessarily unique) function \(f : A \to \mathcal{Q}\) having boundary values \(g\) and of least Dirichlet integral among such functions. Furthermore, each such minimizing \(f\) is Hölder continuous, and \(A \times \mathbb{R}^n \cap \{(x, y) : y \in spt(f(x))\}\) is an \(m\) dimensional real analytic (harmonic) submanifold of \(A \times \mathbb{R}^n\) except possibly for a closed set of Hausdorff dimension not exceeding \(m - 2\).

Assuming that \(m\) and \(n\) and even integers and the usual complex identifications have been made, we show that the \(\mathcal{Q}\) valued function produced by projection mapping slicing of a complex holomorphic chain in \(A \times \mathbb{R}^n\) associated with a \(Q\) fold analytic branched covering of \(A\) is uniquely Dirichlet integral minimizing. Our Hausdorff codimension two singularity estimate for Dirichlet integral minimizing \(\mathcal{Q}\) valued functions is thus the best possible.

**References**

[A] F. Almgren, *Q valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension 2* (being typed, approximately 1500 manuscript pages).


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