EXISTENCE THEOREMS
FOR GENERALIZED KLEIN-GORDON EQUATIONS

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The semilinear elliptic partial differential equation

\[ \text{(1) } Lu = f(x, u), \quad x \in \Omega, \]

is to be considered in smooth unbounded domains \( \Omega \subseteq \mathbb{R}^N, N \geq 2, \)
where

\[ \text{(2) } Lu = - \sum_{i,j=1}^{N} D_i [a_{ij}(x) D_j u] + b(x)u, \quad x \in \Omega, \]

\( D_i = \partial / \partial x_i, \) \( i = 1, \ldots, N; \) each \( a_{ij} \in C^{1+\alpha}_\text{loc}(\Omega), b \in C^\alpha_\text{loc}(\Omega), 0 < \alpha < 1; \) \( b(x) \geq b_0 > 0 \) for all \( x \in \Omega, \)\( L \) is uniformly elliptic in \( \Omega, \) and \( f(x, u) \) satisfies all the conditions in either list (F) or list (F') below. Our main objective is to prove the existence of a positive solution \( u(x) \) of (1) in \( \Omega \) satisfying the boundary condition \( u(x) = 0 \) on \( \partial \Omega \) (void if \( \Omega = \mathbb{R}^N \)), and to obtain asymptotic estimates as \( |x| \rightarrow \infty. \)

The physical importance of the Klein-Gordon prototype

\[ \text{(3) } -\Delta u + b(x)u = \delta [p(x)u^\gamma - q(x)u^\beta], \quad x \in \Omega, \]
arises in particular from nonlinear field theory; the existence of solitary waves and asymptotic behavior as \( |x| \rightarrow \infty \) follow from our theorems. It is assumed in (3) that \( p \) and \( q \) are nonnegative, bounded, and locally Hölder continuous in \( \Omega, \) \( 1 < \gamma < \beta, \) and \( \delta = \pm 1. \) The Hypotheses (F') below are all satisfied if \( \delta = +1 \) and \( p/q \) is bounded and bounded away from zero in \( \Omega. \) Hypotheses (F) are all satisfied if \( \delta = -1, \beta < (N + 2)/(N - 2), N \geq 3, \) and \( q(x) > 0. \)

HYPOTHESES F (UNBOUNDED NONLINEARITY)

(f_1) \( f \in C^\alpha_\text{loc}(\Omega \times \mathbb{R}) \) and \( f(x, t) \) is locally Lipschitz continuous with respect to \( t \) for all \( x \in \Omega. \)

(f_2) There exist positive constants \( s_i > 1 \) and nonnegative, bounded continuous functions \( f_i \in L^2 \Omega, \) \( i = 1, \ldots, I, \) such that

\[ |f(x, t)| \leq \sum_{i=1}^{I} f_i(x) |t|^{s_i}, \quad x \in \Omega, \quad t \in \mathbb{R}, \]

where each \( s_i < (N + 2)/(N - 2) \) if \( N \geq 3. \)

(f_3) \( f(x, t)/t \rightarrow +\infty \) as \( t \rightarrow +\infty \) locally uniformly in \( \Omega. \)

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There exists a positive constant $\epsilon$ such that $(2 + \epsilon)F(x, t) \leq tf(x, t)$ for all $t \geq 0$, $x \in \Omega$, where $F(x, t) = \int_0^t f(x, \tau) d\tau$.

**HYPOTHESES F' (BOUNDED NONLINEARITY)**

1. $(\text{f}^*_1) = (\text{f}_1)$; $(\text{f}^*_2) f(x, 0) = 0$ for all $x \in \Omega$.
2. $(\text{f}^*_3)$ There exists a positive number $T$ such that $f(x, t) < 0$ for all $t > T$ and for all $x \in \Omega$.
3. $(\text{f}^*_4)$ There exists $x_0 \in \Omega$ and $T_0 \in [0, T)$ such that $F(x_0, T_0) > 0$.
4. $(\text{f}^*_5)$ For every bounded domain $M \subset \Omega$ and for every $t_0 > 0$, there corresponds a positive constant $K = K(M, t_0)$ such that $f(x, t) + Kt$ is a non-decreasing function of $t$ on $0 \leq t \leq t_0$ for each fixed $x \in M$.

The unbounded domain $\Omega$ in (1) is allowed to have the general form $\bigcup_{n=1}^{\infty} \Omega_n$, where $\{\Omega_n\}$ is a sequence of smooth bounded domains with $\Omega_n \subset \Omega_{n+1} \subset \Omega$ for $n = 1, 2, \ldots$. For example, $\Omega$ can be an exterior domain, cylindrical or conical domain, or the entire space $\mathbb{R}^N$.

**THEOREM 1.** Suppose that Hypotheses (F) hold and that each $a_{ij}$ and $D_x a_{ij}$ in (2) is bounded in $\Omega \cup \partial \Omega$. Then equation (1) has a positive bounded solution $u(x)$ in $\Omega$ satisfying $u(x) = 0$ identically on $\partial \Omega$ such that $u(x) \to 0$ and $|\nabla u(x)| \to 0$ as $|x| \to \infty$ uniformly in $\Omega$. In the case that $\Omega = \mathbb{R}^N$, (1) has a positive bounded solution $u(x)$ throughout $\mathbb{R}^N$ with this asymptotic behavior at $\infty$.

Specializing to the Schrödinger operator $L = -\Delta + b(x)$, $x \in \Omega$, we prove that the positive solution $u(x)$ in Theorem 1 satisfies

$$\bar{u}(|x|) \leq C \exp(-k|x|), \quad x \in \Omega$$

for some positive constants $C$ and $k$, where $\bar{u}(t)$ denotes the spherical mean square of $u(x)$ over the $(N-1)$-sphere of radius $t$. Sharper estimates arise in the Klein-Gordon case (3) if $b(x) = b$ is a positive constant.

In the case of bounded nonlinearities, we consider boundary value problems of the type

$$\begin{cases}
Lu = \lambda f(x, u), & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega,
\end{cases}$$

in an exterior domain $\Omega$, or in the entire space $\mathbb{R}^N$ with the boundary condition deleted, where $\lambda$ is a positive constant.

**THEOREM 2.** If Hypotheses (F') hold, there exists a positive number $\lambda^*$ such that the boundary value problem (4) has a bounded positive solution $u(x, \lambda)$ in $\Omega \cup \partial \Omega$ for all $\lambda \geq \lambda^*$. The same is true in the case that $\Omega = \mathbb{R}^N$, where the boundary condition in (4) is now deleted.

The theorems below concern the specialization

$$\begin{cases}
-\Delta u + b(x)u = \lambda f(x, u), & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega.
\end{cases}$$
UNIQUENESS THEOREM 3. Suppose in addition to (F') that \( f(x,t)/t \) is bounded in \( \Omega \times (0,T] \) and that \( f(x,t)/t \to 0 \) as \( t \to 0+ \) for all \( x \in \Omega \). Then there exists a positive number \( \lambda_* \) such that the only nonnegative bounded solution \( u(x,\lambda) \) of (5) is identically zero in \( \Omega \) for \( 0 < \lambda \leq \lambda_* \).

THEOREM 4. Suppose in addition to (F') that \( f(x,t)/t \to 0 \) as \( |x| \to \infty \) uniformly in \( 0 < t \leq T \). Then there exists \( \lambda^* > 0 \) such that, for all \( \lambda \geq \lambda^* \), the boundary value problem (5) has a positive solution \( u(x,\lambda) \) in \( \Omega \) satisfying \( u(x,\lambda) \leq C(\lambda) \exp[-\sqrt{b_0/2}|x|] \) for some positive constant \( C(\lambda) \).

We also prove analogous theorems without the uniform positivity hypothesis on \( b(x) \), e.g. \( b(x) \) can be identically zero, or even have negative values.

To prove Theorem 1, we first construct a sequence of solutions \( U_n \) of Dirichlet problems on bounded subdomains \( \Omega_n \) of \( \Omega \), \( n = 1,2,\ldots \), using a variational method of Ambrosetti and Rabinowitz [1]. Let \( u_n \) denote the extension of \( U_n \) to \( \Omega \) defined to be 0 in \( \Omega \setminus \Omega_n \). Using (F) and the variational characterization of \( U_n \) we prove that the sequence of Dirichlet norms \( \|u_n\|_{1,2,\Omega} \) is uniformly bounded and uniformly positive. Then \textit{a priori} estimates, embedding theorems, and a "bootstrap procedure" establish the convergence of a subsequence of \( \{u_n\} \) locally uniformly in \( C^2(\Omega) \) to a solution \( u(x) \) of (1) satisfying \( u(x) = 0 \) identically on \( \partial \Omega \). Furthermore, these techniques imply that there exists a positive constant \( C \), independent of \( x \), such that both

\[
|u(x)| \leq C\|u\|_{1,2,M(x)}, |\nabla u(x)| \leq C\|u\|_{1,2,M(x)}
\]

for all \( x \in \Omega \), where \( M(x) \) denotes a bounded domain for all \( x \in \overline{\Omega} \) with volume of \( M(x) \) constant. The asymptotic behavior of \( u(x) \) stated in Theorem 1 then follows since \( u \in W^{1,2}_0(\Omega) \). This and a comparison argument show that the solution is positive throughout \( \Omega \) and exponentially decaying as \( |x| \to \infty \).

Theorem 1 extends known results of Berestycki and Lions [3], Berestycki, Lions, and Peletier [4], Berger [5], Berger and Schechter [6] and Strauss [10] in three directions: general coefficients (i.e. not necessarily constant or radially symmetric), general domains, and problems with boundary conditions.

We prove Theorem 2 by first constructing subsolutions \( w_n \) of Dirichlet problems in bounded domains \( \Omega_n \), \( n = 1,2,\ldots \), possible for \( \lambda \geq \lambda^* > 0 \) because of a theorem of Rabinowitz [9, p. 177] and a new extension result. Then there exists a sequence of solutions \( u_n \) of (1) in \( \Omega_n \) squeezed between \( w_n \) and the constant supersolution \( T \) by a theorem of Amann [2, p. 283], and \( u_n \) is extended by the definition \( u_n = 0 \) in \( \Omega \setminus \Omega_n \). Following our method in [8] we use \( L^p \)-estimates, Sobolev embedding, and Schauder estimates to prove that \( \|u_n\|_{C^{2+a}(\overline{M})} \) is uniformly bounded with respect to \( n \) for any bounded domain \( M \subset \Omega \). Then a compactness argument shows that a subsequence of \( \{u_n\} \) converges to a bounded positive solution of (4) for \( \lambda \geq \lambda^* \). Theorems 3 and 4 can then be established with the aid of Kato’s \textit{a priori} estimates [7, p. 415].
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