SOME PROBLEMS IN POTENTIAL THEORY
AND THE NOTION OF HARMONIC ENTROPY

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ABSTRACT. Blaschke regions are studied for certain classes of subharmonic functions in connection with the notion of harmonic entropy. A complete description of Riesz measures for some of these classes is obtained. A new analytic inequality is established.

1. Definitions, notations and two basic problems. $k(\rho) (0 \leq \rho < 1)$ will always denote a continuous nonnegative function such that $k(|z|)$ is subharmonic in the open unit disc $D$ (or, equivalently, such that $k(\rho)$ and $\pi k'(\rho)$ are nondecreasing).

DEFINITION 1. Let $\mathcal{M} \subset D$ be a given set, and let $\mathcal{H}_k(\mathcal{M})$ be the set of all nonnegative harmonic functions $u(z)$ in $D$ such that $u(z) \geq k(|z|)$ on $\mathcal{M}$. The following quantity will be called the harmonic $k$-entropy of $\mathcal{M}$:

$$\mathcal{E}(\mathcal{M}; k) = \min\{u(0) : u \in \mathcal{H}_k(\mathcal{M})\}.$$

If $\mathcal{H}_k(\mathcal{M})$ is empty, we set $\mathcal{E}(\mathcal{M}; k) = +\infty$.

DEFINITION 2. $\mathcal{SH}_k$ will denote the class of subharmonic functions $u(z)$ in $D$ such that

$$u(z) \leq C_u k(|z|) \quad (z \in D),$$

where $C_u$ is some constant (depending on $u$).

DEFINITION 3. $\mathcal{A}_k$ will denote the class of analytic functions $f(z)$ in $D$ such that

$$\log |f(z)| \in \mathcal{S}_k.$$

DEFINITION 4. A region $G \subset D$ is called a $k$-Blaschke region if either of two equivalent\(^3\) conditions holds:

(a) for every $u \in \mathcal{SH}_k$

$$b(G; d\mu) = \int_G (1 - |z|) d\mu(z) < \infty,$$

where $d\mu = \Delta u$ is the Riesz measure (i.e. generalized Laplacian) of $u$;

(b) for every $f \in \mathcal{A}_k$

$$\sum_{z_\nu \in G} (1 - |z_\nu|) < \infty,$$

where $\{z_\nu\}$ is the zero set of $f$.

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\(^2\)The use of that term, borrowed from Information Theory, is suggested by this interpretation: if $u(z)$ is conceived as a “signal” of strength $u(0)$ and $k(|z|)$ as the “noise”, then $\mathcal{E}(\mathcal{M}; k)$ is the strength of the weakest signal that overcomes the noise on $\mathcal{M}$.

\(^3\)The equivalence of (a) and (b) is easily proved.
DEFINITION 5. (1) \( S_\zeta \) is the open Stolz angle whose closure is the convex hull of the disk \( |z| < 1/\sqrt{2} \) and the point \( \zeta \in \partial \mathcal{D} \).

(2) For a given closed set \( F \subset \partial \mathcal{D} \), \( G_F \) is the union of \( S_\zeta (\zeta \in F) \).

(3) \( \mathcal{L} \) is the class of regions \( G = \{ z = r e^{i\theta} : 0 \leq r < f(\theta) \leq 1 \} \), where \( f(\theta) \) is a \( 2\pi \)-periodic function satisfying the Lipschitz condition \( |f(\theta_1) - f(\theta_2)| \leq |\theta_1 - \theta_2| \). It is easily seen that every \( G_F \) is an \( \mathcal{L} \)-region.

All results announced below are associated with the following two basic problems.

(A) Given \( k(r) \), characterize regions of finite \( k \)-entropy and find estimates of that quantity.

(B) Given \( k(r) \), characterize \( k \)-Blaschke regions and find effective estimates of the integral (1.3) and the sum (1.3').

The main motivation for (B) is to ultimately obtain a complete description of zero sets for \( \mathcal{A}(k) \)—an objective that we are able to realize only for the case of “slowly increasing” \( k(r) \). Since the problem of \( \mathcal{A}(k) \)-zero sets is essentially a potential-theoretic one, there seems to be no good reason for studying only the special Riesz measures \( d\mu = \Delta \log |f(z)| \) determined by the zeros of an \( f \in \mathcal{A}(k) \), rather than the general Riesz measures for \( \mathcal{S}(k) \). In emphasizing the potential-theoretic, rather than complex-analytic, aspect, we also aim at similar multidimensional problems; in fact, some interesting results [4] for the unit ball in \( \mathbb{R}^m \) have recently been obtained in this circle of ideas (see also §3 below).

Understanding the structure of \( \mathcal{A}(k) \)-zero sets is also an essential first step towards a satisfactory factorization theory for \( \mathcal{A}(k) \); see [1], where the case \( k(r) = \log(1 - r) \) is treated.

As to (A), this problem is instrumental in solving (B). For slowly increasing \( k(r) \) the \( k \)-entropy of an \( \mathcal{L} \)-region \( G \) can be estimated in terms of the following integral

\[
I(G; k) = \int_0^{2\pi} k[f(\theta)] d\theta,
\]

where \( k(1) = k(1^-) \) (= \( \infty \), except for the trivial case of a bounded \( k(r) \)).

In the particular case \( k(r) = (1 - r)^{-\alpha} \) (\( 0 < \alpha < 1 \)) our problems lead to a new elementary inequality (3.3).

2. Slowly increasing \( k(r) \). In this section an additional condition is imposed on \( k(r) \) (\( C \) is some constant):

\[
k(1 - x^2) \leq Ck(1 - x) \quad (0 < x < \frac{1}{2}).
\]

THEOREM 1. (i) \( A \in \mathcal{L} \) is a \( k \)-Blaschke region if and only if \( I(G; k) < \infty \).

(ii) There is a constant \( \lambda > 1 \) depending only on \( k(r) \) with the property that for every \( G \in \mathcal{L} \) there is a \( G' \in \mathcal{L} \), \( G' \supset G \), such that \( I(G', k) < \lambda I(G; k) \) and

\[
\lambda^{-1}I(G; k) \leq \mathcal{E}(G', k) < \lambda I(G; k).
\]

4Detailed proofs will be published elsewhere.
THEOREM 2. The necessary and sufficient condition for a nonnegative Borel measure $d\mu$ in $D$ to be the Riesz measure of a function $u \in \mathcal{H}^{(k)}$ is

\begin{equation}
(2.3) \quad b(G_F; d\mu) \leq CI(G_F; k)
\end{equation}

for all finite sets $F \subset \partial D$ ($C$ is some constant). In this case (2.3) holds also for all $G \in L$, but perhaps with a greater constant $C$.

3. Some other results. (1) A Stolz angle is a Blaschke region for $\mathcal{H}^{(k)}$ if and only if

\begin{equation}
(3.1) \quad \int_0^1 [k(r)(1-r)^{-1}]^{1/2} dr < \infty.
\end{equation}

(See [2].) A similar result for the unit ball in $\mathbb{R}^m$ (with $1/m$ substituted for $1/2$ in (3.1)) has recently been obtained by Krzysztof Samotij (written communication).

(2) Consider the region $G = \{z \in D: M(1-|z|^2)|1-z|^2 > k(|z|)\}$, where $M$ is large enough to ensure that $G \supset S_1$. Then (3.1) implies $I(G; K) < \infty$ and $E(G; k) < \infty$.

(3) A recent result by C. N. Linden [3] shows that, under some extra conditions on the regularity of growth of $k$, (3.1) implies that the above region $G$ is a Blaschke region for $\mathcal{H}^{(k)}$. Similar results describing some “tangential” Blaschke regions for the ball in $\mathbb{R}^m$ are given in [4].

(4) In attempting to extend the results of §2 to wider classes of subharmonic functions, it is natural to consider the particular case $k(r) = (1-r)^{-\alpha}$, where $\alpha$ is fixed, $0 < \alpha < 1$. In this case the assertion (i) of Theorem 1 still holds, provided the function $r = f(\theta)$, which describes the boundary of $G$, has a finite number of maxima and minima. The proof of this depends on

THEOREM 3. There is a constant $C_\alpha$ such that for arbitrary real $x_0 < x_1 < \cdots < x_n$ satisfying

\begin{equation}
(3.2) \quad x_1 - x_0 \leq x_2 - x_1 \leq \cdots \leq x_n - x_{n-1},
\end{equation}

and for arbitrary nonnegative $\{m_i\}_{i=0}^n$, the following inequality holds:

\begin{equation}
\int_{x_0}^{x_n} \left\{ \sum_{i=0}^n m_i(x - x_i)^{-2} \right\}^{\alpha/(\alpha+1)} dx
\leq C_\alpha \left( \sum_{i=0}^n m_i \right)^{\alpha/(\alpha+1)} \left\{ \sum_{i=1}^n (x_i - x_{i-1})^{1-\alpha} \right\}^{1/(\alpha+1)}.
\end{equation}

Because of the restriction (3.2), which cannot be dropped altogether, our results for this case fall short of a complete description of Riesz measures.

REFERENCES


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