THE SULLIVAN CONJECTURE

BY HAYNES MILLER

THEOREM 0. Let $G$ be a locally finite group with classifying space $BG$ and let $X$ be a connected finite dimensional CW complex. Then the space of pointed maps from $BG$ to $X$ is weakly contractible: $\pi_i(\text{map}_*(BG,X)) = 0$ for all $i > 0$. In particular, every map from $BG$ to $X$ is null-homotopic.

A group is locally finite if it is a direct limit of finite groups. When $G$ is of order 2, this theorem represents an affirmative resolution of part of a conjecture of Dennis Sullivan [11]. In this announcement I will sketch a proof of this theorem; details will appear in due course.

Theorem 0 is a straightforward consequence of the following three results.

THEOREM 1. If $X$ is a connected CW complex of finite category and $G$ is a torsion group, then every map $f: BG \to X$ is trivial in $\pi_1$.

THEOREM 2. If $X$ is a nilpotent space such that $H_*(X; \mathbb{F}_p)$ is bounded—that is, $H_i(X; \mathbb{F}_p) = 0$ for all large $i$—then $\text{map}_*(B\mathbb{Z}_p, X)$ is weakly contractible.

THEOREM 3. Let $X$ be a nilpotent space and $G$ a locally finite group. If $\text{map}_*(B\mathbb{Z}_p, X)$ is weakly contractible for every prime $p$ occurring as the order of an element of $G$, then $\text{map}_*(BG, X)$ is weakly contractible.

Theorem 1 is an exercise in $K$-theory and covering spaces. For the remaining theorems, it is convenient to work simplicially; the topological results follow from standard comparison theorems. So “space” will always mean “simplicial set”, usually assumed fibrant.

PROOF OF THEOREM 2. The crux is the following theorem, which distills results of Bousfield and Kan [5], and Dror, Dwyer, and Kan [7].

THEOREM 2.1. Let $W$ be a connected space such that $\overline{H}_*(W; \mathbb{Z}[\frac{1}{p}]) = 0$, and let $X$ be nilpotent. Then the natural map $X \to \mathbb{F}_p\text{-completion}$ induces a weak equivalence $\text{map}_*(W, X) \to \text{map}_*(W, \mathbb{F}_p\text{-completion})$. 

Received by the editors February 2, 1983.

1980 Mathematics Subject Classification. Primary 55S35, 55T10; Secondary 18G55, 13D03, 20J10.

1Partially supported by the Alfred P. Sloan Foundation and NSF grants MCS-8002780 and MCS-8108814(A01).
Let $C$ be the category [4] of unstable right module coalgebras without unit over the mod $p$ Steenrod algebra $A$. Then [4 and 5] show that

$$\text{map}_*(W, \mathbb{F}_{p^n}X) \simeq *$$

provided that

$$\text{Ext}^s_C(\mathbb{H}_*(\Sigma^t W), \mathbb{H}_*(X)) = 0 \text{ for all } t \geq s \geq 0.$$ 

Here $\text{Ext}^s_C(\mathbb{H}_*(\Sigma^t W), -)$ is the $s$th cosimplicially derived functor of $\text{Hom}_C(\mathbb{H}_*(\Sigma^t W), -)$. When $t = 0$, only $\text{Ext}^0_C(\mathbb{H}_*(\Sigma^t W), -)$ is defined, and it is only a pointed set. When $t > 0$, $\mathbb{H}_*(\Sigma^t W)$ is an Abelian cogroup in $C$ and so $\text{Ext}^s_C(\mathbb{H}_*(\Sigma^t W), -)$ is defined for all $s \geq 0$ and is an $\mathbb{F}_p$-vector space. Theorem 2 follows from (2.1) and

**Theorem 2.2.** If $C \in \mathcal{C}$ is bounded then

$$\text{Ext}^s_C(\mathbb{H}_*(\Sigma^t DZ_p), C) = 0 \text{ for all } t \geq s \geq 0.$$

The case $t = 0$ of this theorem follows from the fact that every element of $\mathbb{H}_*(DZ_p)$ is the image of a positive dimensional Steenrod operation. For the remaining cases we invoke a composite functor spectral sequence. Let $N$ be any object of the category $\mathcal{U}$ of unstable right $A$-modules. Then $\Sigma N \in \mathcal{C}$, with trivial diagonal. The module $PC = \ker(\Delta: C \to C \otimes C)$ of primitives is a suspension in $\mathcal{U}$, and $\text{Hom}_C(\Sigma N, C) \cong \text{Hom}_\mathcal{U}(N, \Sigma^{-1}PC)$. We obtain a convergent spectral sequence

$$\text{Ext}^s_\mathcal{U}(N, \Sigma^{-1}R^tP(C)) \Rightarrow \text{Ext}^{s+t}_C(\Sigma N, C).$$

Theorem 2.2 therefore follows from the next two results.

**Theorem 2.3.** If $C$ is a connected bounded $\mathbb{F}_p$-coalgebra then $R^tP(C)$ is bounded for each $t$.

**Theorem 2.4.** If $M \in \mathcal{U}$ is bounded then $\text{Ext}^s_\mathcal{U}(\mathbb{H}_*(\Sigma^nDZ_p), M) = 0$ for all $s, n \geq 0$.

**Proof of Theorem 2.3.** Since any connected coalgebra is a direct limit of finite coalgebras, we may prove (2.3) by establishing suitable results for left derived functors of the functor assigning to a connected commutative $\mathbb{F}_p$-algebra $R$ (henceforth, “algebra”) its module of indecomposables $QR$. As a first approximation we have the following theorem of André [1]:

**Theorem 2.3.1.** If $QR$ is finite, then $L_tQ(R)$ is finite for each $t$.

André’s proof does not seem to yield a bound on the top nonzero dimension, however, so sharper methods are needed to establish (2.3) in general. Quillen [10] tells us to think of $L_tQ(R)$ as the “homology” of the algebra $R$; more generally, any simplicial algebra $X$ admits a surjective weak equivalence $X \to X$ from a cofibrant object [10], and $\pi_*(QX)$ is the “homology” of $X$. Quillen constructs an “Adams spectral sequence”, passing from $\pi_*(QX)$ to $\pi_*(X)$, and Bousfield [3] carries out the coalgebra analogue. We construct a “reverse
Adams spectral sequence”. This spectral sequence is uninteresting when applied to a constant simplicial algebra \( R \). However, “homology commutes with suspension”, and in this setting, the Eilenberg-Mac Lane functor \( \tilde{W} \) is suspension [9]. Let \( D \) be the algebra of higher divided powers, studied at the prime 2 by Dwyer [8] and in general by Bousfield in earlier but unpublished work [2].

\( D \) acts naturally on \( Q\pi_{*}(X) \) for any simplicial algebra \( X \)—in particular, on the vector space of indecomposables in \( \pi_{*}(WR) = \text{Tor}^{R}_{*}(F_{p}, F_{p}) \)—and satisfies a certain “unstable” condition. Let \( \text{Untor}^{D}_{*}(F_{p}, -) \) denote the derived functors of \( F_{p} \otimes_{D} \)—on the category of unstable \( D \)-modules. The spectral sequence then takes the following form.

**Theorem 2.3.2.** There is a natural spectral sequence

\[
E^{2}_{s,t} = \text{Untor}^{D}_{s,t}(F_{p}, Q \text{Tor}^{R}_{*}(F_{p}, F_{p})) \Rightarrow L_{s+t-1}Q(R).
\]

The boundedness result (2.3) follows easily, as does (2.3.1). One also obtains as a corollary a case of an unpublished result of Quillen, namely, that

\[
Q \text{Tor}^{R}_{n}(F_{p}, F_{p}) \simeq L_{n-1}Q(R) \quad \text{for } 1 \leq n \leq 2p-1.
\]

**Proof of Theorem 2.4.** An easy spectral sequence argument shows that we may assume \( n = 0 \). Let \( G(n) \), \( x_{n} \in G(n)_{n} \), corepresent the functor from \( \mathcal{U} \) to sets given by \( M \mapsto M_{n} \): that is, \( \text{Hom}_{A}(G(n), M) \overset{\cong}{\rightarrow} M_{n} \) by \( f \mapsto f(x_{n}) \). \( G(n) \) is evidently projective in \( \mathcal{U} \). We have maps \( G(2n) \to G(2pn) \) by \( x_{2n} \mapsto x_{2pn}P^{n} \) (where \( P^{n} = S^{2n}q \) if \( p = 2 \)). Write \( G_{2n} \) for \( \lim G(2p^{n}) \).

There are natural maps \( G_{2n} \to H_{*}(BZ_{p}) \). Theorem 2.4 for \( n = 0 \) is a direct consequence of the next theorem, due when \( p = 2 \) to Carlsson [6].

**Theorem 2.4.1.** \( \bigoplus_{n=1}^{p-1} G_{2n} \to H_{*}(BZ_{p}) \) is a split epimorphism.

**Proof of Theorem 3.** Let \( G \) be a group, and consider a subcategory \( \mathcal{O} \) of \( \mathcal{O}(G) \), the category of transitive \( G \)-sets and equivariant maps, satisfying

\[ \mathcal{O} \text{ is a full subcategory containing an orbit } T_{0} \]

\[ \text{with a point } x_{0} \text{ whose isotropy subgroup fixes every orbit in } \mathcal{O}. \]

Examples of interest include: (i) Let \( H \) be any subgroup of \( G \) and \( O_{H} \) the category of all those orbits whose isotropy groups are intersections of conjugates of \( H \); (ii) Let \( p \) be a prime and \( O_{p}(G) \) the category of all orbits whose isotropy groups are \( p \)-groups. Let \( T : \mathcal{O} \to (G\text{-sets}) \) be the inclusion functor. Let hocolim be the holim of [5].

**Theorem 3.1.** Let \( \mathcal{O} \) satisfy (*) and let \( E \) be a principal \( G \)-space. There is a natural homotopy equivalence

\[
\text{hocolim}_{\mathcal{O}} E \times_{G} T \simeq E/G.
\]
In particular, if $E$ is contractible then $E \times_O T \simeq BH$ where $H$ is an isotropy group of $T$. We then take the liberty of writing (3.1) as
\[
\underset{\mathcal{O}}{\text{hocolim}} BH \simeq BG.
\]
The resulting spectral sequence [5], for an additive homology theory $h_*$,
\[
L_\ast \underset{\mathcal{O}}{\text{colim}} h_*(BH) \Rightarrow h_*(BG),
\]
would appear to be a useful tool in the study of group cohomology, unifying several well-known constructions.

Theorem 3 results from (3.1), the arithmetic square of [7], and the following consequence of the Sylow theorem.

**Theorem 3.2.** Let $G(p)$ be a $p$-Sylow subgroup of the finite group $G$. Then $H_\ast(\mathcal{O};\mathbb{Z}(p)) = 0$, where $\mathcal{O}$ is (the nerve of) either $\mathcal{O}_p(G)$ or $\mathcal{O}_{G(p)}$.

**Acknowledgments.** I have been helped in this work in specific ways by many mathematicians, among whom I especially want to thank Pete Bousfield, Gunnar Carlsson, Bill Dwyer, John Harper, Dan Kahn, Mark Mahowald, and Bob Thomason.

**References**

2. A. K. Bousfield, *Operations on derived functors of nonadditive functors*, manuscript.