A GENERALIZATION OF TWO CLASSICAL CONVERGENCE TESTS FOR FOURIER SERIES, AND SOME NEW BANACH SPACES OF FUNCTIONS

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ABSTRACT. The norms of these spaces fill the gap between the uniform and the variation norms. Their duals are described in terms of generalized variation. One application of these spaces is a new convergence test for Fourier series which includes both the Dirichlet-Jordan and the Dini-Lipschitz tests [1].

1. The $\kappa$-entropy. $\kappa(s)$ will always denote a nondecreasing concave function on $[0,1]$ such that $\kappa(0) = 0$, $\kappa(1) = 1$; this implies that $\kappa(s)$ is continuous except, perhaps, at $s = 0$.

DEFINITION. Let $E = \{x_1 < x_2 < \cdots < x_n\} \subset [a,b]$ be a finite nonempty set. The following quantity will be called the $\kappa$-entropy of $E$ (relative to $[a,b]$):

$$\kappa(E) = \kappa(E;[a,b]) = \sum_{i=1}^{n+1} \kappa((x_j-x_{j-1})/(b-a)),$$

where $x_0 = a$, $x_{n+1} = b$. For an arbitrary closed set $F \subset [a,b]$ we set

$$\kappa(F) = \kappa(F;[a,b]) = \sup\{\kappa(E): E \subset F \text{ finite}\}.$$

Finally, we set $\kappa(\emptyset) = 0$.

The following properties of the $\kappa$-entropy are easily derived.

(i) $F_1 \subset F_2$ implies $\kappa(F_1) \leq \kappa(F_2)$.

(ii) $\kappa(F_1 \cup F_2) \leq \kappa(F_1) + \kappa(F_2)$.

(iii) If $\text{card } E = n$, then $\kappa(E) \leq (n+1)\kappa(1/(n+1))$; the estimate is sharp and attained for $x_1 - x_0 = x_2 - x_1 = \cdots = x_{n+1} - x_n$.

2. Examples of $\kappa$-entropy.

(a) $\kappa(s) = s$. We have in this case $\kappa(F) = 1 (F \neq \emptyset)$, $\kappa(\emptyset) = 0$.

(b) $\kappa(s) = 1 (0 < s \leq 1)$. Here we have

$$\kappa(F) = \text{card}(F \cup \{a,b\}) - 1 \quad (F \neq \emptyset).$$

(c) $\kappa(s) = s(1 - \log s)$. The corresponding entropy will be denoted by $\kappa_s(F)$ and called the Shannon entropy of $F$ (relative to $[a,b]$).

(d) $\kappa(s) = s^\alpha$. Here $\kappa(F) = \kappa_{\alpha}(F)$ is the Lipschitz entropy $(0 < \alpha < 1)$.

(e) $\kappa(s) = (1 - \frac{1}{2} \log s)^{-1}$; $\kappa(F) = \kappa_{d}(F)$ is the Dini entropy.

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3. The \( \kappa \)-entropy norm.

**Definition.** The \( \kappa \)-entropy norm (or simply the \( \kappa \)-norm) of a real continuous function \( x(t) \) on \([a, b]\) is

\[
\|x\|_\kappa = \|x\|_C + \int_{-\infty}^{\infty} \kappa(E_y, [a, b]) \, dy,
\]

where \( \|x\|_C = \max\{|x(t)| : a \leq t \leq b\} \) and \( E_y = E_y[x] = \{t \in [a, b] : x(t) = y\} \) is the level set of \( x(t) \).

**Examples.** (a) \( \kappa(s) = s \). Here we have \( \|x\|_\kappa = \|x\|_C + \max x(t) - \min x(t) \); thus \( \|x\|_C \leq \|x\|_\kappa \leq 3\|x\|_C \), so that the \( \kappa \)-norm in this case is equivalent to the uniform norm.

(b) \( \kappa(s) = 1 \) (\( 0 < s \leq 1 \)). We have

\[
\|x\| = \|x\|_C + \int_{-\infty}^{\infty} (\text{card } E_y + 1) \, dy = \|x\|_C + M - m + \text{Var } x,
\]

where \( M = \max x(t), m = \min x(t) \); thus

\[
\|x\|_C + \text{Var } x \leq \|x\|_\kappa \leq 3\|x\|_C + \text{Var } x.
\]

(c) The \( \kappa \)-norm corresponding to the Shannon, Lipschitz and Dini entropies is denoted respectively by \( \| \cdot \|_s \), \( \| \cdot \|_\alpha \) and \( \| \cdot \|_d \) and called the Shannon-, Lipschitz- and Dini-entropy norm.

In what follows we assume that \( \kappa(0^+) = 0 \) and \( \kappa(s)/s \to \infty \) (\( s \to 0 \)), since otherwise the \( \kappa \)-norm is equivalent either to the \( C \)-norm or the \( V \)-norm.

4. The spaces \( C_\kappa[a, b] \).

**Theorem 1.** Every \( \kappa \)-norm is homogeneous and convex: \( \|\lambda x\|_\kappa = |\lambda| \|x\|_\kappa \), \( \|x_1 + x_2\|_\kappa \leq \|x_1\|_\kappa + \|x_2\|_\kappa \). Equipped with a \( \kappa \)-norm, the linear set of all real continuous functions \( x(t) \) on \([a, b]\) such that \( \|x\|_\kappa < \infty \) forms a (real) Banach space \( C_\kappa[a, b] \); this space is separable: polynomials are dense in \( C_\kappa[a, b] \).

The homogeneity of the \( \kappa \)-norm follows directly from the definition; however, the proof of the triangle inequality is more difficult.

5. The \( \kappa \)-variation.\(^2\)

**Definition.** The \( \kappa \)-variation of a real function \( \mu(t) \) over \([a, b]\) is

\[
\text{Var}_\kappa \mu = \sup \left\{ \left( \sum_{1}^{n+1} |\mu(x_j) - \mu(x_{j-1})| \right) / \kappa(E; [a, b]) \right\},
\]

where the supremum is taken over all finite sets

\( E = \{x_1 < x_2 < \cdots < x_n\} \subset [a, b] \) and \( x_0 = a, x_{n+1} = b \).

It is easily seen that \( \text{Var}_\kappa \mu < \infty \) implies the existence of unilateral limit values \( \mu(t^+) (a \leq t < b) \) and \( \mu(t^-) (a < t \leq b) \). Every such function \( \mu(t) \) generates a “measure” on the set of all intervals \( I \subset [a, b], \) e.g. \( \mu([\alpha, \beta]) = \mu([\beta^+ - \mu(\alpha^-)), \mu(\alpha, \beta) = \mu([\beta^-) - \mu(\alpha^+) \), and so on. If \( \text{Var}_\kappa \mu < \infty \), then this measure can be extended to all (relatively) open sets \( G \subset [a, b] \) such that

\(^2\)This notion was first introduced in [2] for the Shannon variation (see also [3]).
\( \kappa(\partial G) < \infty \) by the formula \( \mu(G) = \sum_j \mu(I_j) \), where \( I_j \) are the components of \( G \); the series is absolutely convergent. Similarly, for closed sets \( F \subset [a, b] \) we define \( \mu(F) = \mu([a, b]) - \mu([a, b] \setminus F) \).

The linear set consisting of all \( \mu(t) \) \( (a \leq t \leq b) \) such that \( \text{Var}_\kappa \mu < \infty \), provided with the norm \( ||\mu|| = \text{Var}_\kappa \mu \), is a Banach space \( V_\kappa[a, b] \); for the special cases of the Shannon, Lipschitz or Dini variation this space is denoted respectively by \( V_s, V_{l,a} \), and \( V_d \).

6. The \( \kappa \)-integral.

Definition. Let \( x(t) \in C_\kappa[a, b] \) and \( \mu(t) \in V_\kappa[a, b] \). The \( \kappa \)-integral of \( x \) with respect to \( d\mu \) is defined as follows:

\[
\int_a^b x(t) \, d\mu(t) = m \mu([a, b]) + \int_m^M \mu(F_y[x]) \, dy,
\]

where \( m = \min x(t) \), \( M = \max x(t) \), and \( F_y[x] = \{ t \in [a, b] : x(t) \geq y \} \) are the Lebesgue sets of \( x(t) \).

It is easily seen that, by (3) and (4), \( \mu(F_y) \) is summable over \( (m, M) \); we also deduce

\[
\int_a^b x(t) \, d\mu(t) \leq ||x||_\kappa \text{ Var}_\kappa \mu.
\]

If \( \mu \) is of bounded (classical) variation, then \( \int x \, d\mu \) exists as a Riemann-Stieltjes integral and its value coincides with that of the \( \kappa \)-integral.

7. The dual of \( C_\kappa \).

Theorem 2. \( V_\kappa \) is the dual of \( C_\kappa \). This means that every linear functional \( F(x) \) in \( C_\kappa[a, b] \) has the form of a \( \kappa \)-integral (5), where \( \mu \) is uniquely (up to a constant) determined by \( F \). We also have \( \frac{1}{3} \text{ Var}_\kappa \mu \leq ||F|| \leq \text{ Var}_\kappa \mu \).

8. A convergence test for Fourier series. The Dirichlet-Jordan (D-J) convergence test [1] states that the (symmetrical) partial sums \( S_n(t; f) \) of the Fourier series of a \( 2\pi \)-periodic function \( f(t) \) of bounded variation tend to \( \frac{1}{2}(f(t + 0) + f(t - 0)) \) as \( n \to \infty \); if \( f(t) \) is also continuous, then \( S_n(t) \to f(t) \) uniformly.

The Dini-Lipschitz (D-L) test [1] states that \( S_n(t; f) \to f(t) \) uniformly if the modulus of continuity \( \omega(\delta) \) of \( f(t) \) is \( o(|\log \delta|^{-1})(\delta \to 0) \).

The proof of the Dirichlet-Jordan test is based on the C-V duality. However, if instead of the C-V duality we use the Dini-entropy-norm—Dini-variation duality (\( C_d-V_d \)), we obtain a new test that includes both the D-J and the D-L tests.

Definition. A function \( \mu(t) \in V_\kappa[a, b] \) is said to be of vanishing \( \kappa \)-variation at \( t_0 \in [a, b] \) if \( \text{Var}_\kappa \{ \mu(t) - \mu(t_0) \} \chi_\delta(t) \to 0 \) \( (\delta \to 0) \), where \( \chi_\delta(t) \) is the characteristic function of \( [t_0 - \delta, t_0 + \delta] \), and the \( \kappa \)-variation is taken over \( [a, b] \). If this takes place at every point \( t_0 \in [a, b] \), then \( \mu(t) \) is said to be of vanishing \( \kappa \)-variation on \( [a, b] \).

Remark. For the classical variation, if \( f(t) \) is of bounded variation on \( [a, b] \) and continuous at \( t_0 \), then \( f(t) \) is of vanishing variation at \( t_0 \). However, for the \( \kappa \)-variation this is generally not true.
Theorem 3. Let \( f(t) \in V_d[0,2\pi] \) be \( 2\pi \)-periodic and normalized so that 
\( f(t) = \frac{1}{2} [f(t+0) + f(t-0)] \). If \( \varphi(t; t_0) = \frac{1}{2} [f(t_0 + \tau) + f(t_0 - \tau)] \) is of vanishing Dini variation at \( \tau = 0 \), then the Fourier series of \( f(t) \) converges at \( t_0 \) to \( f(t_0) \). If \( f(t) \) is of vanishing Dini-variation on \([0,2\pi]\), then \( S_n(t,f) \to f(t) \) \((n \to \infty)\) uniformly.

A short outline of the proof. We have

\[
S_n(t_0;f) - f(t_0) = \int_0^{\pi} \mathcal{E}_n(t) \, d[\varphi(t; t_0) - f(t_0)],
\]

where

\[
\mathcal{E}_n(t) = \int_t^\pi D_n(\tau) \, d\tau \quad (0 \leq t < \pi), \quad D_n(\tau) = \sin\left( n + \frac{1}{2} \right) \tau \left/ \left( \frac{\tau}{2} \sin \frac{\tau}{2} \right) \right. .
\]

A simple computation shows that \( \mathcal{E}_n \) satisfies

\[
|\mathcal{E}_n(t)| \leq \min\{1, 4((2n+1)t)^{-1}\} \quad (0 \leq t \leq \pi)
\]

and is monotone in each of the intervals \((2k\pi/(2n+1), 2(k+1)\pi/(2n+1))(k = 0,1,\ldots,n-1)\) and \((2n\pi/(2n+1), \pi)\); from this we deduce that the Dini-entropy norms \( ||\mathcal{E}_n||_d \) \((n = 1,2,\ldots)\) are bounded if taken over \([0,\pi]\), and tend to 0 if taken over \([\delta, \pi]\) \((\delta > 0)\). Using this, and (7) and (6), we get the required result.

References


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