ENTROPIES AND FACTORIZATIONS OF TOPOLOGICAL MARKOV SHIFTS

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1. Markov shift entropies. Let $A$ be a nonnegative integral matrix. A well-known construction [7] associates to $A$ a homeomorphism $\sigma_A$ of a totally disconnected compact space called a topological Markov shift, or subshift of finite type. Such Markov shifts play a central role in topological dynamics (see [3]), the investigation of Smale's Axiom A diffeomorphisms [6], and coding theory [1]. We announce here a characterization of the possible values for the topological entropy of such Markov shifts, answering a question raised in [2]. Furthermore, these values possess an arithmetic structure which, together with the isomorphism theorem of Adler and Marcus [2], yields an analogue of prime factorization for Markov shifts up to almost topological conjugacy. Details and applications of these results will appear elsewhere.

We shall always assume $A$ to be aperiodic, i.e. some power of $A$ is strictly positive. The topological entropy of $\sigma_A$ is $\log \lambda$, where $\lambda$ is the spectral radius of $A$ [5]. Perron-Frobenius theory [4] shows that $\lambda$ must be an algebraic integer $> 1$ whose other conjugates have absolute value $< \lambda$. Call an algebraic integer with these properties a Perron number. Our principal result shows these are the only restrictions on Markov shift entropies.

THEOREM 1. If $\lambda$ is a Perron number, then there is a nonnegative aperiodic integral matrix whose spectral radius is $\lambda$.

SKETCH OF PROOF. If $\lambda$ is Perron, let $B$ be the $d \times d$ companion matrix of the minimal polynomial over $\mathbb{Q}$ of $\lambda$. The main difficulty occurs when $B$ has no invariant $d$-sided cones, e.g. when $\text{tr} B < 0$. This is overcome by finding invariant surfaces for $B$ curved towards the dominant eigendirection.

The real Jordan form for $B$ decomposes $\mathbb{R}^d$ into direct sum of the 1-dimensional dominant eigenspace $D = \mathbb{R} w$ for $\lambda$, a collection $\mathcal{E} = \{E\}$ of 1- or 2-dimensional eigenspaces with $\|Bx\| = \gamma_E \|x\| (x \in E)$ for constants $\gamma_E > 1$, and another collection $\mathcal{F} = \{F\}$ of eigenspaces with $\|Bx\| = \gamma_F \|x\| (x \in F)$, $\gamma_F < 1$. If $G = D, E, or F$, let $\pi_G$ be the $B$-equivariant projection from $\mathbb{R}^d$ to $G$. We will use $\pi_D : \mathbb{R}^d \to \mathbb{R} \cong D$ normalized by $\pi_D w = 1$. Put $\pi_C = I - \pi_D$.

Fix $\theta > 0$, and put

$$K_\theta = \{ x \in \mathbb{R}^d : \pi_D x > \theta \| \pi_C x \| \}$$

$$K_\theta(r) = \{ x \in K_\theta : \pi_D x \leq r \}.$$
For sufficiently large $r$, the semigroup generated by $K_{\theta}(r) \cap \mathbb{Z}^d$ contains $K_{2\theta} \cap \mathbb{Z}^d$. Define $\phi: \bigoplus E \rightarrow D$ by

$$\phi\left(\sum_E x_E\right) = \left(\sum_E \|x_E\|^{\log \lambda / \log \gamma_E}\right)_E.$$  

The graph of $\phi$ is $B$-invariant and bowl-shaped since $\log \lambda / \log \gamma_E > 1$. Choose $\xi, \eta > 0$ so that

$$K_{\theta}(r) \subset \left\{ x \in \mathbb{R}^d : \max_{F} \|\pi_F x\| \leq \xi, \pi_D \phi\left(\sum_E \pi_E x\right) \leq \eta \pi_D x \right\} = \Omega.$$  

To construct a nonnegative aperiodic integral matrix $A$ with spectral radius $\lambda$, consider $\Gamma = \{ z \in \Omega \cap \mathbb{Z}^d : \pi_D z \leq s \} = \{ z_j : 1 \leq j \leq n \}$, where $s$ is chosen large enough for (ii) below. Write

$$Bz_i = \sum_{j=1}^{n} a_{ij} z_j$$  

with $a_{ij} \in \mathbb{Z}^+$ using these rules: (i) if $\pi_D z_i \leq s / \lambda$, then $Bz_i = z_{j_0} \in \Gamma$ and let $a_{ij} = \delta_{j_0 j}$; (ii) if $s / \lambda < \pi_D z_i \leq s$, then $Bz_i - z_i \in K_{2\theta}$, and therefore is a nonnegative integral combination of elements of $K_{\theta}(r) \cap \mathbb{Z}^d \subset \Gamma$ and then the $a_{ij}$ can be chosen with $a_{ii} \geq 1$. This yields $A = [a_{ij}]$. If $A$ is reducible, replace $A$ by an irreducible component [4] keeping the same notation. Condition (ii) forces $\text{tr} A > 0$, so $A$ is aperiodic. Using Perron-Frobenius theory, it can be shown that $A$ has spectral radius $\lambda$.

2. An example. Given a Perron number $\lambda$, this proof provides an algorithm for computing a nonnegative aperiodic integral matrix $A$ with spectral radius $\lambda$. When $\lambda$ has negative trace, the dimension of $A$ must be strictly larger than the degree of $\lambda$. For example, the Perron root $\lambda \approx 3.8916$ of $t^3 + 3t^2 - 15t - 46$ has conjugates $\lambda_2 \approx -3.2142, \lambda_3 \approx -3.6775$ and trace $-3$. Using $\eta = 1/10$ in (1), $\Omega$ was searched for a collection $\Gamma$ of lattice points obeying (2). Such a $\Gamma$ with 10 points was found, giving

$$A = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
1 & 4 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 9 & 0 & 0 & 3 & 0 & 0
\end{bmatrix}$$
The characteristic polynomial of $A$ factors over $\mathbb{Q}$ as
\[(t + 1) \times (t^3 + 3t^2 - 15t - 46)(t^6 - 4t^5 - 4t^4 + 27t^3 - 6t^2 - 50t + 24).\]
The roots of the degree 6 irreducible factor are about $0.5134, -1.8277 \pm 0.1641i, 1.9689 \pm 0.6751i,$ and $3.2042$, so the spectral radius of $A$ is indeed $\lambda$.

3. An arithmetic for Perron numbers. Let $P$ denote the set of Perron numbers. Then $P$ is closed under addition and multiplication. If $K$ is a finite extension field of $\mathbb{Q}$, it can be shown that $K \cap P$ is a discrete subset of $[1, \infty)$.

Call $\lambda \in P$ indecomposable if it cannot be written as $\alpha\beta$ with $\alpha, \beta \in P$. Thus 2 is indecomposable; for if $2 = \alpha\beta$ with $\alpha, \beta \notin \mathbb{Z}$, then a conjugate $\beta_i = 2/\alpha_i$ of $\beta$ would have $|\beta_i| = 2/|\alpha_i| > 2/\alpha = \beta$, contradicting $\beta \in P$. A modification due to M. Boyle of this argument proves the following.

**Proposition.** Let $\lambda = \alpha\beta \in P$ with $\alpha, \beta \in P$. Then $\alpha, \beta \in \mathbb{Q}(\lambda)$.

Since $\mathbb{Q}(\lambda) \cap P$ is discrete, it follows that $\lambda$ can be factored into indecomposables, but in only finitely many ways. The Perron factorization of a rational integer coincides with its usual prime factorization, and is unique by the Proposition. Unfortunately, nonuniqueness can occur, as in $(\alpha + 2)^2 = 5\alpha^2$, where $\alpha = (1 + \sqrt{5})/2$, and each factor is indecomposable.

4. Factorizations of topological Markov shifts. Adler and Marcus [2] introduced the notion of almost topological conjugacy, and proved that two aperiodic Markov shifts with the same entropy are almost topologically conjugate. Together with Theorem 1, this proves the following.

**Theorem 2.** Let $\sigma$ be an aperiodic topological Markov shift with entropy $\log \lambda$. Then up to almost topological conjugacy, there is a one-to-one correspondence between factorizations $\sigma = \sigma_1 \times \cdots \times \sigma_n$ of $\sigma$ into a direct product of aperiodic Markov shifts and Perron factorizations $\lambda = \lambda_1 \times \cdots \times \lambda_n$ of $\lambda$, where $\lambda_j \in P$. In particular, the number of such factorizations is finite.

**Corollary 1.** Let $\sigma$ be as in Theorem 2, and assume further that $\lambda$ is indecomposable. Then $\sigma$ is not even almost topologically conjugate to a direct product of nontrivial aperiodic Markov shifts.

Since direct factors of Markov shifts must be sofic, and sofic entropies coincide with Markov shift entropies, we also obtain the following.

**Corollary 2.** Let $p$ be a rational prime. The full $p$-shift cannot be factored into the direct product of homeomorphisms of nontrivial compact spaces.

**References**


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