Invariant Theory of $G_2$

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Introduction. Let $V$ denote $C^n$, and let $G \subseteq \text{SL}(V)$ be a classical subgroup. Then Classical Invariant Theory (CIT) describes the generators and relations of the algebra of invariant polynomial functions $C[mV]^G$, where $m \in \mathbb{Z}^+$ and $mV$ denotes the direct sum of $m$ copies of $V$. Using the symbolic method (see [7]), one can then obtain a handle on the invariants of arbitrary representations of $G$. These classical methods and results have been very useful in many areas of mathematics.

Let $G$ be a connected, simple, and simply connected complex algebraic group. Then $G$ is classical except when $G = \text{Spin}_n$, $n \geq 7$, or in case $G$ is an exceptional group $G_2, F_4, E_6, E_7, \text{or } E_8$. It would be useful to have an analogue of CIT for nonclassical $G$. We have succeeded in establishing an analogue for $G_2$ (described below). We also have a conjectured analogue for $\text{Spin}_7$, but a complete proof requires a computation we are as yet unable to perform.

The Cayley algebra, $G_2$, and the Main Theorem. Let Cay denote the usual (complex) Cayley algebra (see [3]). Then Cay is a nonassociative, noncommutative algebra of dimension 8 over $C$. Let Cay' denote the (7-dimensional) span of all commutators of elements of Cay. Let $\text{tr} : \text{Cay} \rightarrow C$ denote the linear map with kernel Cay' which sends $1 \in \text{Cay}$ to $1 \in C$. Define $x = -x + 2 \text{tr}(x) \cdot 1$, $x \in \text{Cay}$. Then $x \mapsto x$ is an involution such that $xx = n(x) \cdot 1 \in C \cdot 1$ for all $x \in \text{Cay}$. Moreover,

(1) \[ x(xy) = x^2y; \quad (yx)x = yx^2, \quad x, y \in \text{Cay}. \]

(2) \[ x^2 - 2 \text{tr}(x)x + n(x) \cdot 1 = 0, \quad x \in \text{Cay}. \]

(3) \[ x \mapsto n(x) \text{ is a nondegenerate quadratic form on Cay}. \]

The identities in (1), called the alternative laws, are a weak form of associativity. Equation (2) is called the standard quadratic identity.

$G_2$ is the group of algebra automorphisms of Cay. Thus $G_2$ acts trivially on $C \cdot 1$ and faithfully (and orthogonally) on Cay'. From now on, let $G$ denote $G_2$ and let $V$ denote Cay'. By (3), $V$ is $G$-isomorphic to its dual $V^*$.

The following is our main result.
Theorem 4. Let \( m \in \mathbb{Z}^+ \), and let \( x_j \) denote a typical element in the \( j \)th copy of \( V \) in \( mV \), \( 1 \leq j \leq m \).

(4.1) \( C[mV]^G \) is generated by elements \( \text{tr}(x_{i_1} \cdots x_{i_r} \cdots) \), \( r \leq 4 \).

(4.2) The relations of these generators are consequences of identities (1) and (2). Moreover, the relations are generated by ones of degrees 6, 7, and 8 in the \( x_j \).

In (4.1) the elements of \( V = \text{Cay}' \) are multiplied in \( \text{Cay} \); the traces of such products are clearly \( G \)-invariant. The relations of (4.2) are obtained by replacing \( x \) and \( y \) in (1) and (2) by products of elements of \( \text{Cay}' \), multiplying the resulting equations by other products, and then taking traces.

Our results are analogous to those of Kostant-Procesi-Rasmyslev (see [2]) for the adjoint representation of \( \text{SL}_n \). In their case (1) is replaced by associativity, and (2) by the Cayley-Hamilton identity.

As outlined below, we determined generators and relations for \( C[mV]^G \) using techniques of invariant theory and commutative algebra. We then showed, a posteriori, that the generators and relations are as in (4.1) and (4.2). J. Ferrar has informed us that he also has a proof of (4.1).

Generators. We sketch a proof of (4.1): Let \( x_1, \ldots, x_m \) be as in the Theorem. Set

\[
\alpha_{ij} = -\text{tr}(x_ix_j), \quad 1 \leq i, j \leq m, \tag{5.1}
\]

\[
\beta_{ijk} = -\text{tr}(x_i(x_jx_k)), \quad 1 \leq i, j, k \leq m, \tag{5.2}
\]

\[
\gamma_{ijkl} = \text{skew tr}(x_i(x_j(x_kx_l))), \quad 1 \leq i, j, k, l \leq m, \tag{5.3}
\]

where in (5.3) we skew symmetrize in the indices. The invariant \( \alpha_{ij} \) is symmetric in its indices, while \( \beta_{ijk} \) and \( \gamma_{ijkl} \) are skew symmetric in theirs (hence are zero if the same index appears twice).

Let \( \omega \) denote a nonzero element of \( \wedge^3 V \) corresponding to the \( \beta \) type invariants. One can show that wedge multiplication by \( \omega \) gives an isomorphism of \( \wedge^3 V \) with \( \wedge^3 V \), and it follows that generators of \( C[mV]^G \) can be obtained by polarization from those in the case \( m = 4 \). A theorem of Weyl [7, p. 154] says that, when \( m = 4 \), it suffices to consider generators whose degree \( d \) in the fourth copy of \( V \) is at most 1. These generators correspond to invariants in \( C[3V] \) (if \( d = 0 \), and copies of \( V^* \equiv V \) in \( C[3V] \) (if \( d = 1 \)). Using [4] and [5] one can show that \( C[3V]^G \) is generated by \( \alpha \) and \( \beta \) type invariants, and that the covariants in \( C[3V] \) corresponding to the representation \( V \) form a free \( C[3V]^G \)-module with three generators in degree 1, three in degree 2, and one in degree 3. The degree 3 generator corresponds to an invariant of type \( \gamma \); the other generators give nothing new. Thus \( C[mV]^G \) is generated by invariants of types \( \alpha \), \( \beta \), and \( \gamma \); establishing (4.1).

Relations (Proof of (4.2)). Since \( V \cong V^* \), we can just as well consider computing the \( G \)-invariants of the symmetric algebra \( S^*(mV) \). Note that \( \text{GL}_m \) acts naturally on \( S^*(mV)^G \approx S^*(V \otimes \mathbb{C}^m)^G \).

Let \( \phi_i \) denote the standard representation of \( \text{GL}_m \) on \( \wedge^i \mathbb{C}^m \) (so \( \phi_i = 0 \) if \( i > m \)). If \( a_1, \ldots, a_k \in \mathbb{Z}^+ \), let \( \phi = \phi_{a_1} \cdots \phi_{a_k} \) denote the highest weight component of \( S^{a_1}(\phi_1) \otimes \cdots \otimes S^{a_k}(\phi_k) \). If \( a_k > 0 \), we say that \( \phi \) has height \( k \).
The generators of $S^*(V \otimes \mathbb{C}^m)^G$ of (5.1), (5.2), and (5.3) transform by the representations $\phi_1^2$, $\phi_3$, and $\phi_4$, respectively. Thus there is a $GL_m$-equivariant surjection $\pi$ from $R = S^*(\phi_1^2 + \phi_3 + \phi_4)$ to $S = S^*(V \otimes \mathbb{C}^m)^G$, and $I = \ker \pi$ is $GL_m$-invariant and homogeneous (grade $R$ and $S$ in the obvious way). Comparing $R$ and $S$ in degrees $\leq 8$, one finds the following elements of $I$ (see explanation below).

\begin{align}
(6.1) \quad & \phi_1 \phi_5 \subseteq S^2 \phi_3; & \phi_1 \phi_5 \subseteq \phi_1^2 \otimes \phi_4, \\
(6.2) \quad & \phi_2 \phi_5 \subseteq \phi_3 \otimes \phi_4; & \phi_2 \phi_5 \subseteq S^2 \phi_1^2 \otimes \phi_3, \\
(6.3) \quad & \phi_1 \phi_6 \subseteq \phi_3 \otimes \phi_4, \\
(6.4) \quad & \phi_8 \subseteq S^2 \phi_4, \\
(6.5) \quad & \phi_2 \phi_6 \subseteq S^2 \phi_4; & \phi_2 \phi_6 \subseteq S^2 \phi_1^2 \otimes \phi_4, \\
(6.6) \quad & \phi_4^2 \subseteq S^2 \phi_4; & \phi_4^2 \subseteq S^4 \phi_1^2, & \phi_4^2 \subseteq \phi_1^2 \otimes S^2 \phi_3.
\end{align}

Each relation consists of a nontrivial "linear combination" of the given representations in $R$ whose image is zero in $S$. For example, in (6.1), a highest weight vector of the space of relations is $\sigma + \tau$, where

$$
\sigma = \beta_{123}\beta_{145} - \beta_{124}\beta_{135} + \beta_{125}\beta_{134}
$$

is a highest weight vector of $\phi_1 \phi_5 \subseteq S^2 \phi_3$, and

$$
\tau = \alpha_{11}\gamma_{2345} - \alpha_{12}\gamma_{1345} + \alpha_{13}\gamma_{1245} - \alpha_{14}\gamma_{1235} + \alpha_{15}\gamma_{1234}
$$

is a highest weight vector of $\phi_1 \phi_5 \subseteq \phi_1^2 \otimes \phi_4$.

It is not difficult to show that relations (6.1)-(6.6) generate $I$ for any $m$ if they generate in the case $m = 6$ (this has a lot to do with the fact that $\dim V = 7$!), so we may assume $m = 6$. Using techniques of [5] one can show that $S$ has a regular sequence $f_1, \ldots, f_{28}$ consisting of 18 forms of degree 2 and 10 forms of degree 3. Since $S$ is Cohen-Macaulay (even Gorenstein [1, p. 124]) of dimension 28, we find that $S \cong C[f_1, \ldots, f_{28}] \otimes S^0$ (as graded $C[f_1, \ldots, f_{28}]$-module), where $S^0 = S/(f_1, \ldots, f_{28})$ is an artin algebra. Thus the Poincaré series $P(t)$ of $S$ equals $(1 - t^2)^{-18}(1 - t^3)^{-10}P^0(t)$, where $P^0(t) = \sum_{i=0}^l a_it^i$ is the Poincaré series for $S^0$. Since $S$ is Gorenstein, $a_i = a_{l-i}$, $0 \leq i \leq l$, and using a result of Stanley [6] one can show that $l = 24$. Since $S$ has generators of degree $\leq 4$, it follows that $I$ is generated by elements of degree $\leq l + 4 = 28$. Thus we have to show that (6.1)-(6.6) generate $I$ in degrees $\leq 28$. This computation was not easy to do, but was made manageable by the $GL_6$ symmetry and certain estimates arising out of (6.1)-(6.6).

Details are to appear.

REFERENCES


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