A \( p \)-ADIC REGULATOR PROBLEM IN ALGEBRAIC \( K \)-THEORY
AND GROUP COHOMOLOGY

BY J. B. WAGONER

Let \( \mathcal{O} \) be the ring of integers in a number field \( F \). Let \( p \subset \mathcal{O} \) be a prime ideal and \( \mathcal{O}_p = \lim \mathcal{O}/p^n \) be the \( p \)-adic completion of \( \mathcal{O} \). Let

\[
\hat{K}_n(\mathcal{O}) = K_n(\mathcal{O}) \mod \text{torsion},
\]

\[
\hat{K}_n^c(\mathcal{O}_p) = K_n^c(\mathcal{O}_p) \mod \text{torsion},
\]

where \( K_n(\mathcal{O}) \) is the algebraic \( K \)-theory of Quillen [Q] and \( K_n^c(\mathcal{O}_p) = \lim K_n(\mathcal{O}/p^n) \) is the “continuous” or “\( p \)-adic” algebraic \( K \)-theory of \( \mathcal{O}_p \) studied in [W1] by Milgram and the author. Results of [B] and [W1] suggested asking whether

(1) \( \Phi_p : \hat{K}_n(\mathcal{O}) \to \hat{K}_n^c(\mathcal{O}_p) \)

or

(2) \( \Phi : \hat{K}_n(\mathcal{O}) \to \bigoplus_{p | p} \hat{K}_n^c(\mathcal{O}_p) \)

is injective, where \( p \) is a fixed rational prime and \( n > 1 \) is odd. Observe that each \( \Phi_p \) is clearly injective for \( n = 1 \), because \( K_1(\mathcal{O}) = \mathcal{O}^* \) and \( K_1^c(\mathcal{O}_p) = \mathcal{O}_p^* \).

A much harder problem is whether \( \Phi \otimes \mathbb{Z}_p \) is injective. For \( n = 1 \) and \( F \) totally real abelian, injectively of \( \Phi \otimes \mathbb{Z}_p \) on the subgroup of \( \mathcal{O}^* \) consisting of those elements congruent to 1 mod \( p \) for each \( p | p \) is equivalent to nonvanishing of the \( p \)-adic regulator [Br, C]. As an example of (1) let \( F \) be quadratic imaginary. Then is

(3) \( \Phi_p : \mathbb{Z} \cong \hat{K}_3(\mathcal{O}) \to \hat{K}_3^c(\mathcal{O}_p) \cong \mathbb{Z}_p \)

injective when \( p = \text{char}(\mathcal{O}/p) \) is unramified with \( \mathcal{O}_p \cong \mathbb{Z}_p \)? J.-P. Serre asked an equivalent cohomological version of (1) and (2) prior to the circa 1975 \( K \)-theory formulation. For special case (3) injectivity is equivalent to showing \( \Phi_p \otimes \mathbb{Q}_p \) is an isomorphism, which in turn amounts to showing

(4) \( Q_p \cong H^3_c(\text{SL}_n(\mathcal{O}_p); \mathbb{Q}_p) \to H^3(\text{SL}_n(\mathcal{O}); \mathbb{Q}_p) \cong Q_p \)

is an isomorphism for \( n \) large. \( H^3_c \) denotes the continuous cohomology of the \( p \)-adic group \( \text{SL}_n(\mathcal{O}_p) \) and \( H^3 \) is the Eilenberg-Mac Lane cohomology of the discrete group \( \text{SL}_n(\mathcal{O}) \). Compare [L]. Numerous examples of (4)...
result from nonvanishing of the Gross-Coleman $Q_p$-regulator as formulated in [Co]. This regulator connects the $p$-adic dilogarithm and the $L$-function values $L_p(2, \chi \omega^{-1})$.

There is the companion $Z_p$-regulator question to (3): namely, determine the index $R_p$ of

\[ \Phi_p \otimes Z_p : Z_p \approx \tilde{K}_3(\mathcal{O}) \otimes Z_p \rightarrow \tilde{K}_3^c(\mathcal{O}_p) \approx Z_p. \]

For $F = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\mu)$, where $\mu^3 = 1$, we give examples for which $R_p = 1$, i.e., for which $\Phi_p \otimes Z_p$ is an isomorphism. This is done with the aid of a homomorphism $\text{Ch}_p : K_3(\mathcal{O}) \rightarrow \mathbb{Z}/p$ constructed by elementary methods, and the values of $\text{Ch}_p$ turn out to be related experimentally to the values $L_p(2, \chi \omega^{-1}) \mod p$, where $\chi$ is the Dirichlet character of conductor 3 and $\omega$ is the Teichmüller character on $\mathbb{Z}^*$. The details are in [W2].

Aisbett [A] has shown $K_3(\mathbb{Z}/p^n) = \mathbb{Z}/p^{2(n-1)} \oplus \mathbb{Z}/p^2 - 1$ for $p > 2$. To test for examples where (5) is an isomorphism it is sufficient to:

(a) find an explicit element $B \in K_3(\mathcal{O})$,
(b) find an explicit formula for a homomorphism $\text{Ch}_p : K_3(\mathcal{O}) \rightarrow K_3^c(\mathcal{O}_p) \rightarrow K_3(\mathbb{Z}/p^2) \rightarrow \mathbb{Z}/p$,
(c) determine $\text{Ch}_p(B) \neq 0$ in various cases by machine computation.

In the case $F = \mathbb{Q}(\sqrt{-3})$, Tate has shown $K_2(\mathcal{O}) = 0$. Hence $K_3(\mathcal{O}) = H_3(\text{St}_n(\mathcal{O})) = H_3(\text{E}_n(\mathcal{O}))$ for $n$ large enough, where $\text{E}_n(\mathcal{O})$ is the group of $n \times n$ elementary matrices. The class $B \in H_3(\text{E}_n(\mathcal{O}))$ is represented as an explicit sum of 30 simplices in the bar resolution of $\text{E}_3(\mathcal{O})$, and the construction of $B$ makes use of Riley’s hyperbolic representation of the fundamental group of the complement of the figure eight knot [R, M]. As a cohomology class, the homomorphism $\text{Ch}_p : K_3(\mathcal{O}) \rightarrow \mathbb{Z}/p$ comes from the diagram

\[
\begin{array}{ccc}
K_3(\mathcal{O}) & \rightarrow & K_3^c(\mathcal{O}_p) \rightarrow K_3(\mathbb{Z}/p^2) \\
\downarrow & & \downarrow \\
H_3(\text{E}(\mathcal{O})) & \rightarrow & H_3(\text{E}(\mathbb{Z}/p^2)) \approx H_3(\text{E}(\mathbb{Z}/p^2)) \rightarrow \mathbb{Z}/p
\end{array}
\]

The explicit formula for the $E(\mathbb{Z}/p^2)$ invariant cocycle $\text{ch}$ on a three simplex $\sigma[a|b|c]$ in the bar resolution arises from examination of the standard cohomology class $\Delta \in H^2(\text{GL}_n(\mathbb{Z}/p); \text{M}_n(\mathbb{Z}/p))$ of the extension

\[ 0 \rightarrow \text{M}_n(\mathbb{Z}/p) \rightarrow \text{GL}_n(\mathbb{Z}/p^2) \rightarrow \text{GL}_n(\mathbb{Z}/p) \rightarrow 1. \]

The class $\text{ch}$ is a special case of a class constructed in $H^3(\text{GL}_n(A/I^2); I^2/I^3)$ when $A$ is semilocal with radical $I$ such that $A/I$ is finite.

Let $F = \mathbb{Q}(\sqrt{-3})$ and recall [Coh] that a rational prime $p > 3$ is split iff $-3$ is a quadratic residue mod $p$. In this case solve $x^2 \equiv -3 \mod p$. Then $\langle p \rangle = p\overline{p}$, where $p = \langle p, x + \sqrt{-3} \rangle$, $\overline{p} = \langle p, -x + \sqrt{-3} \rangle$, and $\mathcal{O}_p \cong \mathcal{O}_p \cong \mathbb{Z}_p$. 
Theorem. In the following cases $\Phi_p \otimes \mathbb{Z}_p$ is an isomorphism because $\text{Chp}(B) \neq 0$:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$x$</th>
<th>$\text{Chp}(B)$ for $p = (p, x + \sqrt{-3})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>19</td>
<td>4</td>
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<td>67</td>
<td>8</td>
<td>33</td>
</tr>
<tr>
<td>73</td>
<td>17</td>
<td>19</td>
</tr>
</tbody>
</table>

In general we have $\text{Chp}(B) = -\text{Chp}(B)$ so there are nine more cases where $\text{Chp} \neq 0$. The above examples were the only one computed. To make the computation for $\text{Chp}$ we use the isomorphism $i_p: \mathbb{O}/p^2 \rightarrow \mathbb{Z}/p^2$ arising from

$$a + b\nu \rightarrow a + b((3 + 2x - x^2)/4x) \mod p^2,$$

where $\nu = (1 + \sqrt{-3})/2$ and $a, b \in \mathbb{Z}$.

In [Co] Coleman uses the $p$-adic dilogarithm to define a homomorphism $D^*_p: K_3(C_p) \rightarrow C_p$, where $C_p$ is a completion of the algebraic closure of $\mathbb{Q}_p$. When $\mathbb{O}$ is the integers in the number field of $m$th roots of unity, he proves a regulator formula for $K_3(\mathbb{O})$ involving $D^*_p$ and $L_p(2, \chi_\omega^{-1})$, where $\chi$ has conductor $m$ and $\omega$ is the Teichmüller character on $\mathbb{Z}_p^*$. In the case $m = 3$, Theorem 8.1 of [Co] suggests, after simplification, that we should have

$$L_p(2, \chi_\omega^{-1}) \equiv -\tau_B i_p(x) \text{Chp}(B) \mod p$$

for some rational number $\tau_B$ depending only on $B$ and having denominator prime to $p$. The factor $\tau_B$ occurs because we only know $B \in K_3(\mathbb{O})$ is some integer multiple of the generator. Machine computation verifies (6) holds for $\tau_B = 1/18$ in all cases of $p = (p, \pm x + \sqrt{-3})$ considered above.

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Department of Mathematics, University of California, Berkeley, California 94720