BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 10, Number 1, January 1984
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0273-0979/84 $1.00 + $.25 per page

Set theory, An introduction to independence proofs, by Kenneth Kunen, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Company, 1980, xvi + 313 pp., $29.25. ISBN 0-4448-5401-0

This book is intended as a text for beginning graduate courses in axiomatic set theory. It is thoughtfully constructed and very well written and this reviewer has used it successfully for the intended purpose. It could equally well serve as the basis for self-study by mathematicians whose work requires set-theoretic tools or is sensitive to the axioms of set theory.

In such a course of study one first covers the basics, including cardinal and ordinal numbers and the methods of proof and definition by induction. This needs to be done in a formalized setting, based on a system of axioms, if one intends to be precise about foundational matters or to discuss independence results. Thus one also needs to discuss philosophical issues and to give some motivation for the choice of axioms. (Here the system of axioms studied is ZFC—the axioms of Zermelo and Fraenkel, with the Axiom of Choice.) All this is efficiently presented by Professor Kunen in Chapter 1. Although the preface states that he assumed his readers to be familiar (at an undergraduate level) with ordinals and cardinals, the author has done a good job of explaining these basic matters. The graduate students to whom I taught set theory using this book had little trouble there, even though most of them were studying set theory for the first time.

The second general topic in any set theory course of mine is Gödel's universe L of constructible sets **[G1]**. Not only is this a central aspect of axiomatic set theory and an important ingredient in many independence proofs, but its treatment is pedagogically very important, requiring as it does such important technical concepts as absoluteness. Informally, one defines the successive levels L_α of the universe L by induction on the ordinal number α as follows: L_0 is the empty set; $L_{\alpha+1}$ is the collection of subsets of L_α which are first-order definable in the model $(L_\alpha, \in)$, allowing parameters; for limit ordinals λ, L_λ is the union of L_α for all $\alpha < \lambda$. Then L is the union of the levels L_α for all ordinals α. Remarkably $(L, \in)$ satisfies all the ZFC axioms and the generalized continuum hypothesis as well, along with a number of other important mathematical principles. To prove this carefully one needs to bring the definition of L into formalized set theory by giving a treatment within ZFC of first-order definability. Or one can follow the lead of Gödel's monograph **[G2]** and rework the definition of $L_{\alpha+1}$ to remove the motivating idea of definability while smoothing the technical difficulties somewhat. My own choice when teaching a set theory course is to stay close to the informal definition and to beg the technical questions on the grounds that the motivation is an important aid to understanding and that the technicalities can always be returned to later.

In this book the author has tried to give a formal treatment of definability (see §1 of Chapter V), an effort which I do not find completely satisfactory.

Finally a basic course in set theory should present the forcing construction for models of the ZFC axioms. This profound method, the creation of Paul J. Cohen [C], is reaching its maturity, not only in the formal sense that it is now 20 years old, but in that it has become, in the hands of many skillful practitioners, an extremely versatile and powerful tool for proving independence results for the ZFC axioms. One begins with a countable model $(M, \in)$ of these axioms, reduced so that the set M is transitive (for any $x \in M$, $y \in x$ implies $y \in M$). From M one selects a partial ordering P. In the general universe of sets there are certain *generic* subsets G of P, satisfying a technical condition which embodies the idea that G is a representative or nonsingular subset of P, relative to M. (Of course the depth of this idea lies in the details, which will not be given here.) Remarkably it turns out that for such generic sets G, there is a smallest transitive model $(N, \in)$ of the ZFC axioms which satisfies $N \supseteq M \cup \{G\}$. Precisely, one adjoins G to M.

In two chapters (VII and VIII) which are supplemented by many challenging exercises, the author presents a rich variety of examples, of which one could only examine a few in a basic semester course. These include the independence of the Continuum Hypothesis and other cardinal collapsing constructions as well as a treatment of iterated forcing and Martin's Axiom.

The author's attitude toward the Axiom of Choice (AC) is entirely shared by this reviewer: AC is one of the basic axioms of set theory. The independence of AC from the other axioms is therefore, quite reasonably, relegated to the Exercises (E4 in Chapter VII).

The mathematical assertions whose independence is given full treatment here, for example the Continuum Hypothesis or Kurepa's Hypothesis, are statements of an entirely different character, in that they are utterly problematic. No one has the slightest idea whether they are true or false. Moreover, the independence results in set theory show that these are not just open questions of the usual sort. It is an empirical fact that all of mathematics as presently known can be formalized within the ZFC system. It is thus a precise measure of our profound ignorance to know, for example, that the Continuum Hypothesis is independent of these ZFC axioms. Surely this knowledge represents one of the supreme intellectual achievements of this century, even though it also seems to strike yet another blow at human pride.

In covering the details of forcing (i.e. of proving the existence of the forcing extension N and deriving its important properties) the author follows the exposition due to Shoenfield [S]. This is an elegant approach, but it seems to present difficulties for beginners. The author also covers the connection between the forcing construction and Boolean valued models (§7, Chapter VII).

One technical point about the forcing method is usually ignored in expositions of set theory, but it is laudably discussed in this book. According to Gödel's Second Incompleteness Theorem, the existence of a model M of the ZFC axioms cannot be proved in ZFC itself (unless this system is inconsistent). Thus the starting point of the forcing method seems to require an assumption that goes beyond ZFC, and this often confuses sharp-witted beginners. One way out is as follows: introduce M as a formal constant into the language of set theory and let T be the theory in the larger language whose axioms are those of ZFC plus all sentences obtained by relativizing each of the axioms of ZFC to the set M. It follows from the Reflection Principle (which is in turn an important consequence of the Replacement Axiom) that T is a conservative extension of ZFC. That is, any sentence which is provable in T and which does not mention M can already be proved in ZFC; in particular, T is consistent. It is within T that one constructs the forcing extension N of M. In proving that a given assertion φ is true of N, one must identify a finite set of ZFC axioms which one assumes to hold in M and in the universe of sets as well. Then φ is proved within the corresponding finitely axiomatized subtheory of T.

In the course which I taught from this book and have outlined here, my students found it necessary to hop around in the book quite a bit. Although I worried about this, they did not, and their consensus was that the book is demanding but readable. They especially liked the extensive indexing and cross-referencing which the author has provided.

We found no serious mistakes and only a few misprints, most of them easily detected and corrected. One which deserves special mention is that the ordering relation $q \leqslant p$ in part (b) of Definition 2.4 (p. 53), which is fundamental to the forcing construction, should instead be $p \leqslant q$.

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AMERICAN MATHEMATICAL SOCIETY
Volume 10, Number 1, January 1984
© 1984 American Mathematical Society
0273-0979/84 $1.00 + $.25 per page

The representation theory of finite groups, by W. Feit, North-Holland Publishing Co., Amsterdam and New York, 1982, xiv + 502 pp., $55.00 (Dfl. 140.00). ISBN 0-4448-6155-6

The general modular theory of representations of finite groups is the subject of Walter Feit's book, *The representation theory of finite groups*. The theory began with the work of Richard Brauer in the 1940s. Its goals were expressed