[Sto] W. Stoll, *Invariant forms on Grassmann manifolds*, Ann. of Math. Studies, No. 89, Princeton Univ. Press, Princeton, N.J., 1977.

[Ver] D.-N. Verma, *Möbius inversion for the Bruhat ordering on a Weyl group*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 393–398.

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Angular momentum in quantum physics: *Theory and application*, by L. C. Biedenharn and J. D. Louck, Encyclopedia of Mathematics and its Applications, Vol. 8, Addison-Wesley Publishing Company, Reading, Mass., xxix + 716 pp., $54.50. ISBN 0-2011-3507-8

The Racah-Wigner algebra in quantum theory, by L. C. Biedenharn and J. D. Louck, Encyclopedia of Mathematics and its Applications, Vol. 9, Addison-Wesley Publishing Company, Reading, Mass., lxxxviii + 534 pp., $54.50 ISBN 0-2011-3508-6

Angular momentum is a physical quantity which appeared first in classical mechanics. Consider indeed the simplest case of a particle moving in $\mathbf{R}^3$. The observables of this physical system are (C^∞–) functions from $T^*\mathbf{R}^3 \simeq \mathbf{R}^6 \equiv \{(x, p) \mid x, p \in \mathbf{R}^3\}$. The Poisson bracket $\{\cdot,\cdot\}$ associates to every pair (f, g) of such functions the function $\{f, g\}$ defined by

$$\{f, g\} \equiv \sum_{k=1}^{3} \left(\partial_{x^k} f \cdot \partial_{p^k} g - \partial_{x^k} g \cdot \partial_{p^k} f\right). \tag{1}$$

Notice, in particular, that for all $k, l = 1, 2, 3$,

$$\{x^k, x^l\} = 0 = \{p^k, p^l\};\ \{x^k, p^l\} = \delta^{kl} 1. \tag{2}$$

Then the angular momentum $L \equiv x \times p$ is the triple (L^1, L^2, L^3) of functions

$$L^j(x, p) \equiv x^k p^l - x^l p^k, \tag{3}$$

where (j, k, l) is any triple of indices obtained from $(1, 2, 3)$ by cyclic permutations. The Poisson brackets between the components of the angular momentum are

$$\{L^j, L^k\} = L^l, \tag{4}$$

where again (j, k, l) are cyclic permutations of $(1, 2, 3)$.

The angular momentum appeared in the quantum mechanical description of a particle, moving similarly in $\mathbf{R}^3$, as the triple $L = (L^1, L^2, L^3)$ of operators, acting in the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbf{R}^3, d^3x)$, and defined in analogy with (3), by

$$(5) \qquad L^j \equiv Q^k P^l - Q^l P^k,$$

where (j, k, l) are cyclic permutations of $(1,2,3)$; and Q^k, P^l are the selfadjoint operators (formally) defined in $\mathcal{H}$ by

$$(6) \qquad (Q^k\Psi)(x) = x^k\Psi(x); \qquad (P^k\Psi)(x) = -i(\partial_k\Psi)(x).$$

If we now denote by $\{\cdot,\cdot\}$ the "quantum Lie bracket" $[\cdot,\cdot]/i$ (i.e. $\{A, B\} = (AB - BA)/i$) between operators acting on $\mathcal{H}$, the defining relations (6) and (5) lead to

$$(7) \qquad \{Q^k, Q^l\} = 0 = \{P^k, P^l\}; \qquad \{Q^k, P^l\} = \delta^{kl}I$$

for all $k, l = 1,2,3$; and

$$(8) \qquad \{L^j, L^k\} = L^l,$$

where again (j, k, l) are cyclic permutations of $(1,2,3)$. Notice that each operator L^j $(j = 1,2,3)$ can be made to be a selfadjoint operator.

A generalization of the concept of angular momentum—not involving the Q's and P's—was first introduced in quantum physics for the purpose of describing the "spin" of the electron. The objects most likely to turn up in a discussion on the subject with a physicist are what he calls the "Pauli matrices":

$$(9) \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With $S^j \equiv \sigma^j/2$, and $\{\cdot,\cdot\}$ used again to denote the quantum Lie bracket $\{A, B\} = (AB - BA)/i$, we have

$$(10) \qquad \{S^j, S^k\} = S^l$$

for any cyclic permutation (j, k, l) of the indices $(1,2,3)$.

To conclude this preliminary discussion, notice that the three-dimensional, real vector space of all hermitian, trace-zero 2×2 matrices with complex entries, written in the form

$$(11) \qquad \left\{ x \cdot \sigma \equiv \sum_{k=1}^{3} x_k \sigma^k \,|\, x \in \mathbf{R}^3 \right\},$$

is equipped by (10) with precisely the structure of the Lie algebra su(2, **C**); this fact, incidentally, had been recognized and exploited by Cartan in his theory of "spinors".

The central point here is that relations (4), (8) or (10), when taken as the characteristic property of angular momentum, tell us that the study of this physical object is intimately linked to the mathematical theory of the representations of the Lie algebra su(2, **C**), and thus to the theory of representations of the simply connected Lie group SU(2, **C**), which is a double covering of the Lie group O^3_+ of the proper rotations in $\mathbf{R}^3$.

Now, $SU(2, \mathbf{C})$ is not only simply connected, it is compact and simple: a most classical textbook example. Anyone contemplating to write yet another text, specifically devoted to this group and its representations, certainly runs the twin risks of repeating ad nauseam well-known truths, and of indulging in arcane technicalities. That, after a massive 700-page treatise, the putative authors would still have interesting material for a 500-page companion monograph, and that the combined work would make for sustained and stimulating reading, would seem to verge on the impossible.

Yet, the present authors seem to have succeeded; their work is worthy of the ambitious aims of the Encyclopedia. This success can be rationalized in at least two ways. The first pillar on which the work stands should appeal to the teacher of mathematics: $SU(2, \mathbf{C})$ is a prototype on which the general theory of representations of (compact) Lie groups can be explicitly developed, offering a wealth of particularly visualizable illustrations. The second general reason for the interest of the work should appeal to that brand of mathematicians who are inclined to believe that the richest areas of mathematics are those which involve the study of the symmetry properties of geometrical and physical theories; the Erlangen programme is an early example of the former, while the latter is illustrated by the resulting organization of the quantum picture we have formed for the microscopic world, from the theory of elementary particles, nuclei, atoms and molecules, to solid state physics, all of which are illustrated in the books under review.

The development of the general thrust of the books is a testimony to the global organizing power of group theory, and analysis, in physics. On the one hand, the specialist will find a large number of far-reaching applications and much computational help. On the other hand, the pure mathematician, even if he may at times be frustrated by the strong physical bend of the authors, should ponder the dictate, attributed to Hilbert, with which the authors preface their work: "The art of mathematics consists in finding that special case which contains all the germs of generality." Upon working in the field laid out by the authors, any mathematician will, first, reap a rich crop of examples of historical or didactic interest; as fringe benefits, he will also glean precious insights on how physics got to be the way it is, on how it is taught, and perhaps on how the mathematics lectures could be improved to reach more profitably the students in the physical sciences.

Two final remarks should close this review. The first is to draw attention to an "Introduction" by G. W. Mackey, oddly placed in the second volume although it actually covers both volumes (with, in fact, some emphasis on the first); that essay fits all the characteristics of an ideal book review for this Journal... except for its length (some sixty pages, broken in twenty-four sections). The last remark is to mention that these two books contain extensive bibliographies and numerous notes offering historical perspectives and scholarly asides.

These two books are likely to remain unchallenged reference material for years to come.

GÉRARD G. EMCH