LOCAL RINGS OF FINITE SIMPLICIAL DIMENSION

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In this note $R$ denotes a (noetherian, commutative) local ring with residue field $k$. Our purpose is to determine those $R$, over which $k$ has finite (co-)homological dimension as an $R$-algebra in the simplicial theory of André [1] and Quillen [11]. Recall that regular local rings are characterized in this theory by the vanishing of the homology group $D_2(k|R)$. Furthermore, it is known that each of the conditions (i) $D_3(k|R) = 0$, (ii) $D_4(k|R) = 0$, (iii) $D_q(k|R) = 0$ for $q \geq 3$, is equivalent to $R$ being a complete intersection, by which we mean that in some (hence in any) Cohen presentation of the completion $\hat{R}$ as a homomorphic image of a regular local ring $\hat{R}$, the ideal $\ker(\hat{R} \to \hat{R})$ is generated by an $\hat{R}$-regular sequence.

**Theorem 1.** If $D_q(k|R) = 0$ for $q$ sufficiently large, then $R$ is a local complete intersection.

**Remark 1.** The previous statement proves a conjecture of Quillen [11, Conjecture 11.7] and answers a question of André [1, p. 118]. When $\text{char}(k) = 0$, its validity is established by [11, Theorem 7.3] and Gulliksen’s result in [10].

**Remark 2.** It has been shown by the author and Halperin [4] that in characteristic zero the conclusion of the theorem holds under the (much) weaker assumption that $D_q(k|R) = 0$ for infinitely many values of $q$. It is not known whether the restriction is essential, and in fact it is an open question, in any characteristic, whether the cotangent complex is rigid, i.e.: Does $D_q(k|R) = 0$ for a single $q \geq 1$ imply that $R$ is a complete intersection?

The proof of Theorem 1 uses some precise information on the growth of the coefficients of the formal power series $P_R(t) = \sum_{n \geq 0} \dim_k \text{Tor}_i^R(k,k)t^n$. For our present purpose it is best expressed in terms of the radius of convergence $r(P_R(t))$. Note that the inequality $r(P_R(t)) > 0$ has been known for a long time to hold for any local ring $R$, and that for complete intersections one even has $r(P_R(t)) \geq 1$.

**Theorem 2.** The inequality $r(P_R(t)) \geq 1$ characterizes complete intersections.

**Remark 3.** The last result has been conjectured both by Golod and by Gulliksen, and proved, in case $R = \bigoplus_{i \geq 0} R_i$ is graded with $R_0 = k$ a field of characteristic zero, by Felix and Thomas [9]. Results related to Theorem 2 are discussed in [2]; complete proofs will appear in [3].

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PROOF OF THEOREM 1. Denote by $L_{k|R}$ the cotangent complex of the $R$-algebra $k$ (so that $H_*(L_{k|R}) = D_*(k|R)$ by definition), and by $S^k$ the symmetric algebra functor, extended—dimensionwise—to simplicial $k$-vector spaces. There is a convergent "fundamental spectral sequence", due to Quillen [11, Theorem 6.3], such that

$$E_{p,q} = H_{p+q}(S^k_{L_{k|R}}) \Rightarrow \text{Tor}^R_{p+q}(k,k).$$

With $E(t)$ denoting the formal power series $\sum_{j \geq 0} \left( \sum_{p+q=j} \dim_k E_{p,q} \right) t^j$, this implies a coefficientwise inequality $E(t) \geq P_R(t)$; hence for the radii of convergence one obtains

$$r(P_R(t)) \geq r(E(t)).$$

The simplicial vector space $L_{k|R}$ decomposes, according to Dold [7], in a direct sum $V \oplus (\bigoplus W_i)$ of simplicial vector spaces, such that $H_*(V) = 0$, $H_n(W_i) \simeq k$ for some integer $n_i$, and $H_j(W_i) = 0$ for $j \neq n_i$. Since, by the general results of [1 and 11], $D_q(k|R)$ is finite dimensional for each $q$, our assumption implies the direct sum above involves only finitely many spaces $W_1, \ldots, W_m$. By [7] again,

$$H_*(S^k L_{k|R}) = H_*(S^k V) \otimes \bigotimes_{i=1}^m H_*(S^k W_i).$$

According to Dold and Thom [8], $H_*(S^k V) = k$, and

$$H_*(S^k W_i) \simeq H_*(K(Z, n), k),$$

where $K(Z, n)$ denotes, as always, the Eilenberg-Mac Lane space whose unique nontrivial homotopy group is infinite cyclic and located in degree $n$. Setting

$$\vartheta(n, k)(t) = \sum_{j \geq 0} \dim_k H_j(K(Z, n), k)t^j,$$

one can write

$$E(t) = \prod_{i=1}^m \vartheta(n_i, k)(t).$$

The circle $S^1$ being a familiar $K(Z, 1)$, and the complex projective space $CP^\infty$ being a $K(Z, 2)$, one has, over any field $k$,

$$\vartheta(1, k) = (1 + t), \quad \vartheta(2, k) = (1 - t^2)^{-1}.$$  

Furthermore, the identity

$$\vartheta(n, k) = (1 + (-1)^{n+1} t^n)^{(-1)^{n+1}}$$

is valid for all $n \geq 1$, when char($k$) = 0.

Finally, when char($k$) = $p > 0$, one has

$$r(\vartheta(n, k)(t)) = 1 \quad \text{for } n \geq 3.$$  

For $p = 2$ this is established by Serre as a consequence of his computation of the mod 2 cohomology of $K(Z, n)$: cf. [12, §3, Theorem 5 and Corollary 1 to Theorem 2]. When $p$ is odd, one can use in a similar way Cartan's
isomorphism \( H_*(K(Z,n),k) \cong \Gamma_*(C_*) \), where \( \Gamma \) denotes the free algebra with divided powers, and \( C_* \) is a graded vector space, determined in [6]. More precisely, according to [6, Main Theorem and Theorem 3], \( c_j = \dim C_j \) can be described as being the number of solutions of the equations

\[
h_1 + 2 \sum_{i=2}^{s} p^{i-1} h_i + 2 \sum_{i=1}^{s-1} p^{i-1} u_i = j, \quad h_1 + 2 \sum_{i=2}^{s} h_i + \sum_{i=1}^{s-1} u_i = n
\]

in nonnegative integers \( s, h_i, u_i \), subjected to the conditions \( u_i \leq 1; h_i + u_i \geq 1 \) for \( i = 1, 2, \ldots, s - 1; h_s \geq 1 \). In particular, \( c_j \) does not exceed the number of decompositions of \( j \) as a sum of \( 2n - 1 \) nonnegative integers, hence

\[
C(t) = \sum_{j \geq 0} c_j t^j \leq (1 - t)^{-2n+1},
\]

yielding the inequality \( r(C(t)) \geq 1 \). It can be replaced by an equality since \( C(t) \) has integral coefficients and is not a polynomial. Because of this last circumstance one can also apply a result of Babenko [5], according to which \( r(C(t)) = r(\vartheta(n,k)(t)) \), hence (5) holds.

Formulas (1)–(5) show that Theorem 1 is a consequence of Theorem 2.

REFERENCES


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