Let $H^{s,p}$ denote the Sobolev space with norm

$$\|u\|_{s,p} = \|F(1 + |\xi|^2)^{s/2}F u\|_p,$$

where $F$ denotes the Fourier transform, $F$ its inverse and $\|w\|_p$ is the $L^p(\mathbb{R}^n)$ norm. In many linear and nonlinear problems one comes across the question:

For which functions $V(x)$ does there exist an estimate of the form

$$\|Vu\|_q \leq C\|u\|_{s,p}, \quad u \in H^{s,p}?$$

In this note we find a large class of functions $V$ for which (2) holds. To do this we introduce a new family of norms $M_{a,r,t,\delta}(V)$ for $0 \leq \alpha$, $0 < \delta < 1$, $1 < r < \infty$ and $1 < t < \infty$. For $x \in \mathbb{R}^n$ let

$\omega_{\alpha}(x) = |x|^a - n$, \hspace{1cm} $0 < \alpha < n$,

$= 1 - \log |x|$, \hspace{1cm} $\alpha = n$,

$= 1$, \hspace{1cm} $n < \alpha$.

When $0 < \alpha$ we define

$$M_{a,r,t,\delta}(V) = \left( \int \left( \int_{|x-y|<\delta} |V(x)|^r \omega_{\alpha}(x-y) \, dx \right)^{t/r} \, dy \right)^{1/t}, \quad 1 \leq t < \infty,$$

$$= \sup_y \left( \int_{|x-y|<\delta} |V(x)|^r \omega_{\alpha}(x-y) \, dx \right)^{1/r}, \quad t = \infty.$$

For $\alpha = 0$ we put

$$M_{0,r,t,\delta}(V) = \|V\|_t.$$

We also set

$$M_{a,r,t}(V) = M_{a,r,t,1}(V)$$

and make the following basic assumption.

HYPOTHESIS A. The parameters $\alpha$, $r$, $s$, $t$, $p$, $q$ satisfy:

A1. $0 \leq \alpha, s$; $1 \leq q \leq r < \infty$; $1 \leq p < \infty$; $1 \leq t \leq \infty$.

A2. $1/q \leq 1/p + 1/t$.

A3. $\alpha/nr \leq s/n + 1/q - 1/p - 1/t$.

A4. We do not have both $q = t$ and $n = sp$ or both $p = 1$ and $s/n = 1/t + 1/q$.

Let $M_{a,r,t}$ denote the set of those functions $V(x)$ such that $M_{a,r,t}(V) < \infty$.  

Received by the editors July 13, 1983.

1980 Mathematics Subject Classification. Primary 47B38, 46E35; Secondary 26D10, 35J60, 45P05, 46E30, 47H17.

Key words and phrases. Function spaces, multiplication operators, estimates.
THEOREM 1. Under Hypothesis (A) assume that:
(a) If \( q < r \) and \( s < n \), then either \( p \neq 1 \) or inequality A3 is strict.
(b) If \( q = r \) and \( s < n \), then either \( r < t \leq p' \) or inequality A3 is strict.

Let \( V(x) \) be a function in \( M_{\alpha,r,t} \). Then multiplication by \( V \) is a bounded operator from \( H^s,p \) to \( L^q \). There is a constant \( C_0 \) depending only on the parameters such that
\[
(3) \quad \| Vu \|_q \leq C_0 M_{\alpha,r,t}(V) \| u \|_{s,p}, \quad u \in H^s,p.
\]
Moreover, there are constants \( C_1, C_2 \) depending only on the parameters such that
\[
(4) \quad \| Vu \|_q \leq C_1 M_{\alpha,r,t,\delta}(V) \| u \|_{s,p} + C_2 M_{\alpha,r,t}(V) \| u \|_p, \quad u \in H^s,p,
\]
and \( C_1 \) does not depend on \( \delta \).

COROLLARY 2. If, in addition,
\[
(5) \quad M_{\alpha,r,t,\delta}(V) \to 0 \quad \text{as} \quad \delta \to 0
\]
then for every \( \epsilon > 0 \) there is a constant \( K \) such that
\[
(6) \quad \| Vu \|_q \leq \epsilon \| u \|_{s,p} + K \| u \|_p, \quad u \in H^s,p.
\]
If \( t \neq \infty \), then multiplication by \( V \) is a compact operator from \( H^s,p \) to \( L^q \). The same conclusion holds when \( t = \infty \) if
\[
(7) \quad \int_{|x-y|<1} |V(x)|^r \omega_\alpha(x-y) \, dx \to 0 \quad \text{as} \quad |y| \to \infty.
\]

THEOREM 3. Let \( \psi(\rho) \) be a positive function such that \( \int_0^1 \psi(\rho) \rho^{-1} \, d\rho < \infty \). Under Hypothesis A, inequalities (3) and (4) hold without the restrictions (a) and (b) of Theorem 1 provided we replace \( \omega_\alpha(x) \) by \( \omega_\alpha(x) \psi(x)^{-b} \) in the definition of \( M_{\alpha,r,t}(V) \), where \( b = r(1 + 1/q - 1/p - 1/r - 1/t) \).

We apply these results to a nonlinear problem. Let \( m \) be a positive integer, \( \Omega \) an arbitrary (bounded or unbounded) domain \( \mathbb{R}^n \) and let \( W = H^m,2(\Omega) \) be the completion of \( C_0^\infty(\Omega) \) in \( H^m,2 \). Let \( a(u,v) \) be a hermitian bilinear form on \( W \) satisfying
\[
K_1^{-2} \| u \|_{m,2}^2 \leq a(u) = a(u,u) \leq C^2 \| u \|_{m,2}^2, \quad u \in W.
\]
Let \( f(x,v) \), \( g(x,v) \) be continuous functions on \( \Omega \times \mathbb{R} \) such that \( f(x,v) = \partial g(x,v) / \partial v \). We assume that
\[
(8) \quad g(x,v) \leq B(x,v) = \sum_{k=1}^N V_k(x) \| u \|_{q,k}^*, \quad x \in \Omega, \quad v \in \mathbb{R},
\]
and
\[
M_{\alpha_k,r_k,t_k,\delta}(V_k) \to 0 \quad \text{as} \quad \delta \to 0, \quad 1 \leq k \leq N,
\]
where
\[
1 < q_k/2 + 1/t_k, \quad \alpha_k/nr_k \leq mq_k/n + 1 - q_k/2 - 1/t_k.
\]
If \( t_k = \infty \), we assume that (7) holds for \( V_k, \alpha_k, r_k \). By Theorem 1 there are constants \( M_k \) such that

\[
\int_{\Omega} B(x, u(x)) \, dx \leq \sum_{k=1}^{N} M_k \|u\|_{m,2}^{q_k}, \quad u \in W.
\]

Assume that

\[
K_2 = \int_{\Omega} g(x, 0) \, dx
\]

exists, and put

\[
M(R) = R^{-2} \left( \sum_{k=1}^{N} M_k (K_1 R)^{q_k} - K_2 \right), \quad \lambda_0^{-1} = \inf_{0 < R} M(R).
\]

Let \( A \) be the operator associated with \( a(u, v) \) (cf. [8]), and let \( \lambda > 0 \). We are looking for a solution of

\[
Au = \lambda f(x, u).
\]

**Theorem 4.** If \( \lambda < \lambda_0 \), then, for any \( R \) such that \( \lambda M(R) < 1 \), (10) has a solution satisfying \( a(u) \leq R^2 \).

There is a connection between the spaces \( M_{\alpha, r, t} \) and the Lorentz spaces \( L^{\sigma, t} \) (for the definitions cf. [3, 11, 13]).

**Theorem 5.** If \( 0 < 1/\sigma - 1/t = \alpha/\nu r, r \leq t < \infty \), then

\[
M_{\alpha, r, t}(V) \leq C\|V\|_{L^{\sigma, t}}, \quad V \in L^{\sigma, t}.
\]

If we combine this with Theorem 1 we obtain

**Theorem 6.** If \( t < \infty \) and \( 1/q - 1/p \leq 1/t \leq 1/\sigma < 1/q, 1/\sigma + 1/p \leq s/n + 1/q \), then

\[
\|Vu\|_q \leq C\|V\|_{L^{\sigma, t}} \|u\|_{s, p}.
\]

Special cases of inequality (2) were proved by Stummel [12], Balslev [2], Berger-Schechter [4] and Schechter [7, 8, 9]. Our solution of (10) avoids some of the hypotheses of Noussair-Swanson [6]. The suggestion that there should be an inequality of the form (12) is due to H. Brezis.

Theorem 1 is proved by using Bessel potentials as investigated by Aronszajn-Smith [1]. Inequality (3) is equivalent to

\[
\left| (G_s * f, Vu) \right| \leq C\|f\|_p \|v\|_q \times \left( \int \left( \int |V(x)|^{t/r} G_s(x-y)^{\phi(x-y)^{-r}} \, dx \right)^{t/\tau} \, dy \right)^{1/t}
\]

holding for a suitably chosen function \( \phi \). This is derived by tedious and tricky estimates. Corollary 2 follows by standard arguments and Theorem 3 is a slight variation. Theorem 4 is proved by variational techniques using (8) and (9). Theorem 5 is proved by using the fact that \( H^{s,p} \subset L^q \) for certain values of \( s, p, q \). By real interpolation we find that \( H^{s,p} \subset L^q \) for specific values. This leads to (11). Theorem 6 is merely a combination of Theorems 1 and 5.
REFERENCES


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