THE ASYMPTOTIC BEHAVIOR
OF NONLINEAR SCHRÖDINGER EQUATIONS

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We consider the nonlinear Schrödinger equation with power interactions
\[(NS) \quad i\partial u/\partial t = -\frac{\lambda}{2} u + \lambda|u|^{p-1} u \]

in $\mathbb{R}^n$, $n \geq 2$, with $\lambda > 0$. Proposing a new method for studying the large time behavior of the solutions of (NS), we prove the following theorem. $H_0 = -\frac{\lambda}{2} \Delta$ is the free Hamiltonian and

\[ \Sigma = \{ u \in L^2(\mathbb{R}^n); \| u \|_2 + \| \nabla u \|_2 + \| xu \|_2 < \infty \}, \]

where $\| u \|_q$ denotes the $L^q$-norm of $u$.

**Theorem.** Let $1 + 2/n < p < 1 + 4/(n - 2)$. Then for any $u_0 \in \Sigma$ there exists a unique $u_+ \in L^2(\mathbb{R}^n)$ such that the solution $u(t)$ of (NS) with $u(0) = u_0$ has the free asymptote $u_+$ as $t \to \pm \infty$:

\[ (2) \quad \lim_{t \to \pm \infty} \| u(t) - \exp(-i t H_0) u_+ \|_2 = 0. \]

**Remark.** Since it is shown by Glassey [4] and Strauss [6] that if $1 < p \leq 1 + 2/n$ any nontrivial solution $u(t)$ of (NS) with $u(0) \in \Sigma$ never satisfies (2), our theorem achieves the least possible exponent $1 + 2/n$ for this direction.

In the sequel we shall prove the theorem. Our proof is based on the following observation: Since the asymptotic profile of the free evolution $\exp(-i t H_0)f$ is given by $(1/it)^{n/2} \exp(ix^2/2t)f(x/t)$ and (NS) is transformed by the conjugation $C$,

\[ (3) \quad u(t, x) = (Cv)(t, x) = (1/it)^{n/2} \exp(ix^2/2t)v(1/t, x/t), \]

into the new equation
\[(TNS) \quad i\partial v/\partial t = -\frac{\lambda}{2} \Delta v + \lambda|t|^{n(p-1)/2-2} |v|^{p-1} v, \]

the relation (2) is equivalent to the existence of

\[ (4) \quad \lim_{t \to \pm 0} v(t) \equiv v_+(0) \quad \text{in} \quad L^2(\mathbb{R}^n). \]

Here and hereafter $\hat{f}$ and $\check{f}$ are the Fourier transform of $f$ and the inverse Fourier transform of $f$, respectively. The equation (TNS) has almost the same form as (NS) and, for $p > 1 + 2/n$, $t^{n(p-1)/2-2}$ is integrable near $t = 0$. Thus we expect the existence of the limit (4) for those $p$'s.

The equation (NS) has interested many authors and there is quite a body of literature. Among them, we mention the following which are related to
our result. For $1 \leq p < 1 + 4/(n-2)$, the global existence of the solution $u(t)$ of (NS) with $u(0) \in H^1(\mathbb{R}^n)$ is proved by Ginibre and Velo [1]. In [2] they also show the above theorem for $1 + 4/n < p < 1 + 4/(n-2)$ (see also Lin and Strauss [5]). The lower exponent $1 + 4/n$ is subsequently decreased to

$$\gamma(n) = (n + 2 + \sqrt{n^2 + 12n + 4})/2n$$

in Strauss [7], but the allowed $u(0)$ are restricted to be small in a certain norm.

**Proof.** From [1] and [2] we already know that (NS) has a unique global solution $u(t, \cdot) \in C(\mathbb{R}^1; \Sigma)$ with $u(0) = u$. We note that the solution of (NS) means the so-called mild solution of the integral equation associated with the differential equation (NS) (see [1]). Then a direct computation shows that $v(t) = (C^{-1}u)(t) \in C(\mathbb{R}^1; \Sigma)$ is a unique solution of (TNS).

We prove the theorem for $t \to +\infty$ with $1 + 2/n < p \leq 1 + 4/n$ only. The other cases may be proved similarly. We first obtain two conservation laws for (TNS). We multiply (TNS) by $t^{2-n(p-1)/2} \partial v/\partial t$ and take the real part. This leads us to

$$i^{2-n(p-1)/2} \|\nabla v(t)\|^2 + \frac{4}{p+1} \int_{\mathbb{R}^n} |v(t,x)|^{p+1} dx$$

(5)

$$\geq i^{2-n(p-1)/2} \|\nabla v(s)\|^2 + \frac{4}{p+1} \int_{\mathbb{R}^n} |v(s,x)|^{p+1} dx$$

for all $0 < s < t < +\infty$. We note that this rather formal calculation can be easily justified by the regularizing technique of Ginibre and Velo [1]. Next we multiply (TNS) by $v$ and take the imaginary part to obtain

$$\|v(t)\|_2 = \|v(s)\|_2, \quad 0 < s < t < +\infty.$$  

(6)

By (5) and (6) we conclude that

$$t^{2-n(p-1)/2} \|\nabla v(t)\|^2 < C_1, \quad \|v(t)\|_{p+1} < C_2, \quad \|v(t)\|_2 < C_3,$$

for all $t \in (0, 1]$, where $C_1, C_2$ and $C_3$ depend only on $\|v(1)\|_{H^1}$ and $\|v(1)\|_{p+1}$. Let $\varphi \in H^1(\mathbb{R}^n)$. By (TNS),

$$\left(v(t) - v(s), \varphi\right) = \int_s^t \left(\frac{\partial v(\tau)}{\partial \tau}, \varphi\right) d\tau$$

$$= -\frac{i}{2} \int_s^t (\nabla v(\tau), \nabla \varphi) t\tau$$

$$- i \int_s^t t^{n(p-1)/2-2} |v(\tau)|^{p-1} v(\tau, \varphi) d\tau$$

for $0 < t, s < +\infty$, where $(\cdot, \cdot)$ is the inner product in $L^2(\mathbb{R}^n)$. Since $n(p-1)/2 - 2 > -1$ for $p > 1 + 2/n$ and $H^1(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, (7) and (8) show that the weak limit

$$\lim_{t \to +00} v(t) \equiv v(0)$$

exists in $L^2(\mathbb{R}^n)$. Now choose $\varphi = v(t)$ in (8). Then

$$|v(t) - v(s), v(t)| \leq \frac{1}{2} \int_s^t \|\nabla v(\tau)\|_2 d\tau \cdot \|\nabla v(\tau)\|_2$$

$$+ \int_s^t t^{n(p-1)/2-2} |v(\tau)|^{p-1} d\tau \cdot \|v(t)\|_{p+1},$$

(10)
for all $0 < s < t < +\infty$. Applying (7) to (10), we have

$$
|\langle v(t) - v(s), v(t) \rangle| \leq C_4 \left[ \frac{4}{n(p-1)} \left\{ t^{n(p-1)/2-1} - s^{n(p-1)/4} t^{n(p-1)/4-1} \right\} \right.
$$

$$
+ \frac{2}{n(p-1)-2} \left( t^{n(p-1)/2-1} - s^{n(p-1)/2-1} \right).$$

Let $s \to +0$ and use (9) to obtain

$$
|\langle v(t) - v(0), v(t) \rangle| \leq C_5 t^{n(p-1)/2-1}
$$

with $C_5 > 0$ depending only on $n$, $p$, $\|v(1)\|_{p+1}$ and $\|v(1)\|_{H^1}$. Therefore,

$$
\|v(t) - v(0)\|^2 = (v(t) - v(0), v(t)) - (v(t) - v(0), v(0))
$$

$$
\leq C_5 t^{n(p-1)/2-1} + |\langle v(t) - v(0), v(0) \rangle| 
$$

$$
\to 0 \quad (t \to +0).
$$

Returning to (NS) we see that

$$
\|\exp(-itH_0)\bar{v}(0) - u(t)\|_2 \to 0 \quad (t \to +\infty),
$$

as desired. \qed

The construction of wave operators and the asymptotic completeness problem will be discussed elsewhere.

REFERENCES


