1. Introduction. As described in the survey article [6], the study of “incomplete polynomials”, as introduced by G. G. Lorentz [4] in 1976, leads to results on the asymptotic properties of polynomials orthogonal on an infinite interval (cf. [5]) and to theorems on the convergence of “ray sequences” of Padé approximants for Stieltjes functions. Here we present a generalization of the theory for incomplete polynomials which unifies many of the previous results. The essential question which serves as the starting point for the investigation is the following:¹

Suppose \( w(x) \) is a nonnegative weight function continuous on its support \( \Sigma \subset \mathbb{R} = (-\infty, \infty) \). (By the support of \( w \) we mean the closure of the set where \( w \) is positive.) Assume that \( w(x) \) vanishes at points of \( \Sigma \); that is, \( Z := \{ x \in \Sigma : w(x) = 0 \} \neq \emptyset \) (or, in case \( \Sigma \) is unbounded, then \( |x|w(x) \to 0 \) as \( |x| \to \infty \)). If \( P_n \) is an arbitrary polynomial of degree at most \( n \), then the sup norm over \( \Sigma \) of the weighted polynomial \( [w(x)]^n P_n(x) \) actually “lives” on some compact set \( S \subset \Sigma - Z \) which is independent of \( n \) and \( P_n \). The question is to determine the smallest such set \( S \).

For example, if \( w(x) = x^\theta/(1-x) \) with \( \Sigma = [0, 1] \), \( 0 < \theta < 1 \), then, as shown in [2, 8], \( S \) is the subinterval \([0^{2}, 1]\).

In this paper we use potential theoretic methods to show how \( S \) can be obtained for a class of weight functions. The assumptions on \( w \) are given in

DEFINITION 1.1. Let \( w : \mathbb{R} \to [0, +\infty) \). We say that \( w \) is an admissible weight function if each of the following properties holds:

(i) \( \Sigma := \text{supp}(w) \) has positive capacity.
(ii) The restriction of \( w \) to \( \Sigma \) is continuous on \( \Sigma \).
(iii) The set \( Z := \{ x \in \Sigma : w(x) = 0 \} \) has capacity zero.
(iv) If \( \Sigma \) is unbounded, then \( |x|w(x) \to 0 \) as \( |x| \to \infty \), \( x \in \Sigma \).

Here, and throughout the paper, the term “capacity” means inner logarithmic capacity (cf. [10, p. 55]). For any set \( E \subset \mathbb{R}^2 \), its capacity will be denoted by \( C(E) \). If \( K \) is a compact set with positive capacity, then \( \nu_K \) denotes the unique unit equilibrium measure on \( K \) with the property that (cf. [10, p. 60])

\[
\int_K \log|x-t| d\nu_K(t) = \log C(K)
\]

quasi-everywhere (q.e.) on \( K \). (A property is said to hold q.e. on a set \( A \) if the subset \( E \) of \( A \) where it does not hold satisfies \( C(E) = 0 \).)
For an admissible weight \( w \), we always set
\[
Q(x) := \log\left(\frac{1}{w(x)}\right).
\]
Finally, if \( K \subset \Sigma - Z \) is compact and \( C(K) > 0 \), we define the F-functional of \( K \) by the formula
\[
F(K) := \log C(K) - \int_K Q \, dv_K.
\]

The theorems of §2 show that, for a class of weight functions, \( S \) is derived by maximizing the F-functional. Also, if \( \pi_m \) denotes the collection of all polynomials of degree at most \( m \) and \( \| \cdot \|_A \) denotes the sup norm over a set \( A \), we describe the asymptotic behavior of the errors in the weighted Chebyshev problem
\[
E_n(w) := \inf \{ \| [w(x)]^n \{ x^n - p_{n-1}(x) \} \|_{\Sigma} : p_{n-1} \in \pi_{n-1} \},
\]
for positive integers \( n \), as well as asymptotic properties (as \( n \to \infty \)) of the extremal polynomials \( T_n(x; w) = x^n + \cdots + \infty \) which satisfy
\[
E_n(w) = \| [w(x)]^n T_n(x; w) \|_{\Sigma},
\]
as well as asymptotic properties (as \( n \to \infty \)) of the extremal polynomials \( T_n(x; w) = x^n + \cdots + \infty \) which satisfy
\[
E_n(w) = \inf \{ \| [w(x)]^n \{ x^n - p_{n-1}(x) \} \|_{\Sigma} : p_{n-1} \in \pi_{n-1} \},
\]
for positive integers \( n \).

2. Statements of main results.

**THEOREM 2.1.** Let \( w \) be an admissible weight function with support \( \Sigma \). Then there exists a compact set \( S \subset \Sigma - Z \) with \( C(S) > 0 \) that has the following properties.

(a) For every compact set \( K \subset \Sigma - Z \) with \( C(K) > 0 \),
\[
F(K) \leq F(S),
\]
where \( F \) is defined in (1.3).

(b) If equality holds in (2.1), then \( S \subset K \).

(c) For any positive integer \( n \), if \( P_n \in \pi_n \) and the inequality
\[
\| [w(x)]^n P_n(x) \| \leq M \quad (M = \text{constant})
\]
holds q.e. on \( S \), then it holds q.e. on \( \Sigma \).

(d) The errors \( E_n(w) \) defined in (1.4) satisfy
\[
[E_n(w)]^{1/n} \geq \exp(F(S)), \quad \forall n = 1, 2, \ldots.
\]

Clearly properties (a) and (b) uniquely determine the set \( S = S(w) \) of Theorem 2.1. In the special case when \( w(x) \equiv 1 \) on \( \Sigma \) and \( \Sigma \) is compact, then \( S \) is just the support of the equilibrium measure \( \nu_{\Sigma} \) for \( \Sigma \).

Of practical importance is the characterization of \( S \) given in

**THEOREM 2.2.** Assume that, in Theorem 2.1, the set \( \Sigma - Z \) is the finite union of disjoint nondegenerate intervals and that \( Q(x) \) of (1.2) is convex in each of the components of \( \Sigma - Z \). Then the following additional properties hold.

(a) The compact set \( S \) of Theorem 2.1 is the finite union of nondegenerate disjoint closed intervals, at most one in each component of \( \Sigma - Z \).
(b) Equality holds in (2.1) if and only if $S \subset K$ and $C(K - S) = 0$.

(c) For any positive integer $n$, if $P_n \in \pi_n$, then

$$\|w(x)^n P_n(x)\|_\Sigma = \|w(x)^n P_n(x)\|_S.$$  

(d) The errors $E_n(w)$ of (1.4) satisfy

$$\lim_{n \to \infty} |E_n(w)|^{1/n} = \exp(F(S)).$$  

The proof of Theorem 2.1 follows by showing that $S$ is actually the support of a measure which solves an extremal problem for generalized energy integrals, as we now describe. Let $\mathcal{M}(\Sigma)$ denote the collection of all positive unit Borel measures $\mu$ with supp$(\mu) \subset \Sigma$, and define

$$I_w[\mu] := \int \int [\log |x - t| - Q(x) - Q(t)] d\mu(x) d\mu(t)$$

for $\mu \in \mathcal{M}(\Sigma)$. Following methods of Frostman (cf. [10]) we obtain

**Theorem 2.3.** Let $w$ be an admissible weight function with support $\Sigma$ and let

$$V_w := \sup\{I_w[\mu] : \mu \in \mathcal{M}(\Sigma)\}.$$  

Then there exists a unique measure $\mu_w \in \mathcal{M}(\Sigma)$ such that $I_w[\mu_w] = V_w$. Moreover, $S_w := \text{supp}(\mu_w)$ satisfies all the properties stated in Theorem 2.1; that is, $S_w = S$.

Concerning the limiting distribution of the zeros of the extremal polynomials $T_n(x; w)$ we have

**Theorem 2.4.** With the assumptions of Theorem 2.2, let $\{x_{k,n}\}_{k=1}^n$ denote the zeros of the extremal polynomial $T_n(x; w)$ of (1.5), and let $\nu_n$ be the associated unit Borel measure defined by

$$\nu_n(B) := \left(\frac{1}{n}\right)\{|k : x_{k,n} \in B|\}.$$  

Then, in the weak star topology,

$$\lim_{n \to \infty} \nu_n = \mu_w,$$

where $\mu_w$ is the extremal measure of Theorem 2.3. Furthermore,

$$\lim_{n \to \infty} |T_n(x; w)|^{1/n} = \exp\left(\int \log |z - t| d\mu_w(t)\right)$$

uniformly on every compact set of the plane disjoint from the convex hull $[\lambda, \tau]$ of $S$.

**3. Applications.** For Jacobi weights of the form $w(x) = x^\theta(1-x)^\eta$, $0 < \theta < 1, \Sigma = [0, 1]$, or $w(x) = (1 - x)^{\lambda_1}(1 + x)^{\lambda_2}$, $\lambda_1, \lambda_2 > 0, \Sigma = [-1, 1]$, maximizing the associated $F$-functional leads to the results of [2, 9, 3 and 7] concerning incomplete polynomials.

For a weight $W$ on $\mathbb{R}$ of the form $W(x) = \exp(-q(x))$, where $q(x)$ is even and convex on $\mathbb{R}$ and $q(x)/\ln x \to \infty$ as $x \to \infty$, we can also analyze the extremal problems

$$e_n(W) := \inf\{\|W(x)\{x^n - p_{n-1}(x)\}\|_\mathbb{R} : p_{n-1} \in \pi_{n-1}\}, \quad n = 1, 2, \ldots,$$
and the corresponding extremal polynomials \( t_n(x; W) = x^n + \cdots + \pi_n \) satisfying \( e_n(W) = ||W(x)t_n(x; W)||_R \). After maximizing the appropriate \( F \)-functional, Theorem 2.2(c) yields

\[
(3.2) \quad ||W(x)P_n(x)||_R = ||W(x)P_n(x)||_{[-a_n,a_n]}, \quad \forall P_n \in \pi_n,
\]

where \( a = a_n \) is a root of the equation

\[
(3.3) \quad n = \frac{2}{\pi} \int_0^1 \frac{axq'(ax)}{\sqrt{1-x^2}} \, dx.
\]

Letting \( w_n(x) := \exp(-q(a_n x)/n) \) with \( \Sigma_n := \text{supp}(w_n) = [-1, 1] \), it follows from (3.2) that

\[
e_n(W) = a_n^n E_n(w_n), \quad t_n(a_n x; W) = a_n^n T_n(x; w_n).
\]

If the weights \( w_n \) converge uniformly to an admissible weight \( w \) on \([-1, 1]\), it can be shown that the asymptotic behaviors (as \( n \to \infty \)) of \( E_n(w_n) \) and \( T_n(x; w_n) \) are the same as that for \( E_n(w) \) and \( T_n(x; w) \). These facts lead to the results of [5] for \( W(x) = \exp(-|x|^\alpha), \alpha \geq 1 \), as well as to \( L^\infty \)-analogue of the \( L^2 \)-results in [1] for \( W(x) = \exp(-\exp |x|) \).

**REFERENCES**


**DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, LOS ANGELES, CALIFORNIA 90032**

**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FLORIDA, 33620**