SIMPLE CLOSED GEODESICS ON $H^+/\Gamma(3)$
ARISE FROM THE MARKOV SPECTRUM

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1. Let

$$H^+ = \{z = x + iy : y > 0\}$$

be the complex upper half-plane, and let

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n}; \ a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}$$

be the principal congruence subgroup of level $n$ in the modular group $SL(2, \mathbb{Z}) = \Gamma(1)$. In this note we are concerned with $\Gamma(3)$. Let $S$ be the Riemann surface $H^+/\Gamma(3)$ and let $\pi : H^+ \to S$ be the projection map. $S$ is a sphere with four punctures.

A hyperbolic element $\gamma$ is a Möbius transformation of $H^+$ that has two real fixed points; its axis $A_\gamma$ is the circle with center on $\mathbb{R}$ connecting the fixed points. Write $\xi_\gamma, \xi'_\gamma$ for the fixed points of $\gamma$. If $\gamma \in \Gamma(3)$ is hyperbolic, $A_\gamma$ projects to a closed geodesic on $S$; conversely, every closed geodesic on $S$ arises in this way. A simple closed geodesic is one that does not intersect itself.

The Markov Spectrum will be described in detail in §2. Here we note the definition of the Markov function $M(\theta)$. For real irrational $\theta$ set

$$M(\theta) = \sup\{c > 0 : |\theta - p/q| < 1/cq^2 \text{ for infinitely many reduced fractions } p/q\}.$$  \hspace{1cm} (1.1)

In the range $M(\theta) < 3$, $M$ assumes only a denumerably infinite set of values $M_\nu \uparrow 3$. The numbers $M_\nu$ constitute the Markov Spectrum, which we denote by MS.

The connection between simple closed geodesics on $S$ and MS is established in the following way. For $\beta \in \Gamma(3)$ write $A_\gamma \cap \beta A_\gamma$ to mean $A_\gamma \cap \beta A_\gamma \neq \emptyset, A_\gamma$, i.e., the intersection is a single point in $H^+$. The following criterion is easy to prove:

$$\pi(A_\gamma) \text{ is nonsimple if and only if } A_\gamma \cap \beta A_\gamma \text{ for some } \beta \in \Gamma(3) - \langle \gamma \rangle.$$  \hspace{1cm} (1.2)

But in this statement we know nothing about $\beta$ except that if is not elliptic ($\Gamma(3)$ contains no elliptic elements).

**THEOREM 1.** If $\pi(A_\gamma)$ is nonsimple, there is a parabolic element $P$ in $\Gamma(3)$ such that $A_\gamma \cap PA_\gamma$.

Theorem 1 leads directly to the main result:
THEOREM 2. Let $\gamma \in \Gamma(3)$ be hyperbolic. Then $\pi(A_\gamma)$ is simple if and only if $M(\xi_\gamma) = M(\xi'_\gamma) < 3$.

Since Zagier [3] has recently given an asymptotic formula describing this portion of MS, we can deduce from Theorem 2:

COROLLARY 1. Let $N_S(T)$ be the number of simple closed geodesics on $S$ of hyperbolic length $\leq T$. Then $T \ll N_S(T) \ll T^2$.

The implied constants are effective. This inequality contrasts with results on $N(T)$, the number of closed geodesics of length $\leq T$, first obtained by H. Huber [1].

Returning to Theorem 1, we make use in the proof of the following known facts:

(1.3) A simple loop $L$ contained in $\pi(A_\gamma)$ cannot bound a disk, i.e., each component of $S - L$ must contain at least one puncture.

(1.4) If $L$ bounds a disk with exactly one puncture, then $L$ determines a conjugacy class of parabolic elements of $\Gamma(3)$.

We remark that $\pi(A_7)$ has a finite number of self-intersections, since it is a real-analytic curve.

Now it can be shown that $\pi(A_\gamma)$, assumed nonsimple, contains a simple loop $L$ surrounding a single puncture $p$. We are indebted to A. F. Beardon for a proof of this fact that is shorter and simpler than the original one.

There is a lift of $\pi(A_\gamma)$ lying on $A_1$ and starting from a point $\xi$, i.e., the lift is an interval $(\xi, \gamma\xi)$ of $A_1$. Using (1.4) one can show that there is a parabolic element $P$ in $\Gamma(3)$ such that the lift of $L$ is an interval $(\xi_0, P\xi_0) \subset (\xi, \gamma\xi)$. Thus $A_\gamma \cap PA_\gamma$, as asserted.

Full details will follow in a paper written jointly with A. F. Beardon. This paper also contains the following result. Let $T$ be a finitely generated fuchsian group and let $S = H^+ / \Gamma$ be the associated Riemann surface. Then Theorem 1 holds for $S$ if and only if $S$ is of genus zero and has either three or four punctures or deleted disks.

2. We now return to the Markov Spectrum (MS); for a fuller account see [2, pp. 29–32]. In (1.1) and the following lines we defined MS to be the set of values $\{M_\nu\}$ assumed by the Markov function $M(\theta)$ in the range $M(\theta) < 3$. In order to calculate $M_\nu$ we introduce Markov triples. A triple of positive integers $(x, y, z)$ is called a Markov triple if $x^2 + y^2 + z^2 = 3xyz$, $1 \leq x \leq y \leq z$. The first triples are $(1,1,1), (1,1,2), (1,2,5), \ldots$, and the rest can be recursively generated. Order the triples by the size of $z$ so that $1 = z_1 \leq 2 = z_2 \leq \cdots \leq z_\nu \cdots$. With each triple $(x_\nu, y_\nu, z_\nu)$ there is associated a pair of real quadratic conjugates

\begin{equation}
(2.1) \quad \theta_\nu, \theta'_\nu = \frac{1}{2} + y_\nu / x_\nu z_\nu \pm \frac{1}{2} (9 - 4/z_\nu^2)^{1/2}, \quad \nu \geq 1.
\end{equation}

The connection of $M(\theta)$ with $\theta_\nu$ is that

\begin{equation}
(2.2) \quad M(\theta_\nu) = M_\nu = |\theta_\nu - \theta'_\nu| = (9 - 4/z_\nu^2)^{1/2}.
\end{equation}

We have $M_1 = 5^{1/2}, M_2 = 8^{1/2}, M_3 = (221)^{1/2}/5, \ldots, \to 3$.

Next, introduce the equivalence relation:
\[ \theta \sim \psi \text{ if and only if } \psi = (a\theta + b)/(c\theta + d) \text{ with integers } a, b, c, d \quad (2.3) \]
and \( ad - bc = \pm 1. \)

Then \( \theta \sim \psi \text{ if and only if } \]
\[ M(\theta) = M(\psi). \quad (2.4) \]
Moreover, the regular continued fraction expansions of \( \theta \) and \( \psi \) agree from a certain point on. Also
\[ M(\theta) < 3 \Rightarrow \theta \sim \theta_\nu \quad \text{for some } \nu \geq 1. \quad (2.5) \]

Indeed, the definition of MS shows that \( M(\theta) = M_\nu = M(\theta_\nu), \text{ so } \theta \sim \theta_\nu \text{ by } (2.4). \)

The numbers \( \{\theta_\nu\}, \{\theta'_\nu\}, \) together with their equivalents under (2.3), are called Markov quadratic irrationalities (MQI). Theorem 2 may now be restated.

**Theorem 2'**. \( \pi(A_\gamma) \) is simple if and only if \( \xi_\gamma \) is equivalent to a MQI.

We can associate MS to hyperbolic elements of \( \Gamma(3) \). For each \( \nu \) there is a \( \gamma_\nu \in \Gamma(3) \) whose fixed points are \( \xi_{\gamma_\nu} = \theta_\nu, \xi'_{\gamma_\nu} = \theta'_\nu. \) Namely, dropping the subscript \( \nu, \) let \( \zeta = 1 \) if \( z \) is odd, otherwise \( \zeta = 1/2. \)

Define
\[ B = \begin{pmatrix} (N + x(2y + xz)\zeta M)^{-1} & (2x^2 z - 4xy + z)\zeta M \\ x^2 z \zeta M & (N - x(2y + xz)\zeta M)^{-1} \end{pmatrix}, \]
where \( M > 0 \) is the smallest integral solution of the Pell equation
\[ x^4 (9z^2 - 4)\zeta^2 M^2 + 4 = N^2. \]

Then it can be shown that \( B \) is the \( \Gamma(1) \)-primitive matrix fixing \( \xi, \xi'. \) Moreover, \( B \in \Gamma(3) \) if \( 3|M, \) otherwise \( B^2 \in \Gamma(3). \) But the first case never occurs, so \( B^2 \) is the \( \Gamma(3) \)-primitive matrix fixing \( \xi, \xi'. \)

By abuse of notation we say \( \gamma \in \text{MS} \) if \( \gamma \in \Gamma(3) \) and \( \xi_{\gamma} \sim \theta_\nu \text{ for some } \nu \geq 1. \) If \( \gamma \in \text{MS} \) so does \( \gamma V^{-1}, V \in \Gamma(1), \) since \( \gamma V^{-1} \in \Gamma(3) \) by normality of \( \Gamma(3) \) in \( \Gamma(1) \) and \( \xi_{\gamma V^{-1}} = V \xi_{\gamma} \sim \gamma V \theta_\nu \sim \theta_\nu. \) That is,
\[ (2.7) \text{ the conjugacy class of } \gamma \text{ in } \Gamma(1) \text{ belongs to MS if } \gamma \in \text{MS}. \]

We now prove Theorem 2. Suppose \( \pi(A_\gamma) \) is nonsimple; then by Theorem 1 there is a \( \delta \) conjugate to \( \gamma \) in \( \Gamma(3) \) for which \( A_\delta \wedge S^3 A_\delta, \text{i.e., } |\xi_\delta - \xi'_{\delta}| > 3. \) By a translation in \( \Gamma(1) \) we may assume \( -1 < \xi'_{\delta} < 0; \text{ then } \xi_\delta > \xi'_{\delta} + 3 > 1. \) Thus \( \xi_\delta \) is "reduced" [4, p. 73] and the regular continued fraction of \( \xi_\delta \) is pure periodic; also \( \xi'_{\delta}. \)

Let \( \xi_\delta = (b_0, b_1, \ldots, b_{k-1}) \) for \( k \geq 1; \) then \( -1/\xi'_{\delta} = (b_{k-1}, \ldots, b_0) \) [4, p. 76]. Here \( b_{nk+\nu} = b_\nu \) for \( 0 \leq \nu < k, n \geq 0. \) Set
\[ m_\mu = (b_{\mu}, b_{\mu+1}, \ldots, b_{\mu+k-1}) + (0, b_{\mu-1}, b_{\mu-2}, \ldots, b_{\mu-k}), \quad \mu \geq k. \]

By periodicity \( m_\mu = m_{\mu+k}. \) Moreover, \( M(\xi_\delta) = \lim_{\mu \to \infty} m_\mu \) [2, p. 29].

Therefore, for all \( \epsilon > 0 \) and \( n > N, \)
\[ 3 < \xi_\delta - \xi'_{\delta} = m_k = m_{nk} < \lim_{\mu \to \infty} m_\mu + \epsilon < M(\xi_\delta) + \epsilon, \]

implying
\[ M(\xi_\gamma) = M(\xi_\delta) > 3, \]
as asserted.
Conversely, assume $\pi(A_\gamma)$ is simple. Then certainly $|\xi_\gamma - \xi'_\gamma| \leq 3$, otherwise $A_\gamma \wedge S^3 A_\gamma$. Since $\pi(A_\gamma)$ is simple if and only if $\pi(V A_\gamma) = \pi(A_{V^{-1}V'})$ is simple for all $V \in \Gamma(1)$—because $\Gamma(3) < \Gamma(1)$—we have

\[ |V \xi_\gamma - V \xi'_\gamma| \leq 3, \quad V \in \Gamma(1). \]

Assuming $\gamma \notin \text{MS}$ we shall produce a $V \in \Gamma(1)$ that contradicts ($\ast$).

At this point we observe that $M(\xi_\gamma) \neq 3$ for any $\gamma \in \Gamma(1)$. Indeed, $\xi_\gamma$ is a quadratic irrationality and $M(\theta)$ is never 3 if $\theta$ is a quadratic irrational [2, p. 32]. It follows that $\gamma \notin \text{MS}$ implies $M(\xi_\gamma) > 3$, that is,

\[ |\xi_\gamma - p_n/q_n| < 1/(3 + h)q_n^2, \quad (p_n, q_n) = 1, \]

for some $h > 0$, on a sequence $q_n \to \infty$. Write $V_n = (q'_n, -p'_n : q_n, -p_n) \in \Gamma(1)$. Then with $\xi_\gamma = \xi$, $\xi'_\gamma = \xi'$,

\[
|V_n \xi - V_n \xi'| = q_n^2 |\xi - p_n/q_n| |\xi' - p_n/q_n| > (3 + h)|\xi - \xi'| \\
\geq \frac{|\xi' - \xi| + |\xi - p_n/q_n|}{3 + h} > \frac{1 + 1/3q_n^2|\xi - \xi'|}{3 + h} > 3,
\]

for $n \geq n_0$. For $V = V_{n_0}$ we have a contradiction to ($\ast$).

We close with a comment on Corollary 1. The existence of long simple geodesics on $H^+ / \Gamma(3)$ is not hard to prove topologically. The feature of Corollary 1 is that the lengths are known explicitly: they are

\[ \text{length } A_{B^2_\nu} = 2 \log \frac{t_\nu + \sqrt{t_\nu^2 - 4}}{2}, \quad t_\nu = \text{trace } B^2_\nu. \]

REFERENCES


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