CHARACTERIZING \( k \)-DIMENSIONAL UNIVERSAL MENGER COMPACTA

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The disjoint \( k \)-cells property \((DD^k P)\), isolated by J. W. Cannon \([Ca]\), has played the critical role in the characterization theorems for finite-dimensional manifolds \((R. D. Edwards [Ed], F. Quinn [Qu])\) and for manifolds modeled on the Hilbert cube \((H. Toruńczyk [To])\). A metric space \( X \) has \( DD^k P \) if each pair \( f, g: I^k \to X \) of maps of a \( k \)-cell into \( X \) can be arbitrarily closely approximated by maps with disjoint images.

By Toruńczyk’s characterization theorem, a compact AR is homeomorphic to the Hilbert cube \( Q \) iff it satisfies \( DD^k P \) for \( k = 0, 1, 2, \ldots \).

On the other hand, the Cantor set \( C = \mathbb{I}^0 \) is the only 0-dimensional compactum that satisfies \( DD^0 P \) (i.e. does not have isolated points). From R. D. Anderson’s characterization of the universal curve \( \mu^1 [An] \) (the 1-dimensional Peano continuum with no local cut points which does not contain a nonempty open set that can be embedded into the plane), it follows that \( \mu^1 \) is the only connected \((C^0)\), locally connected \((LC^0)\) 1-dimensional compactum that satisfies \( DD^1 P \). The construction of the universal curve generalizes to give the \( k \)-dimensional universal Menger space \( \mu^k \):

Subdivide \([0, 1]^{2k+1} = A_1 \) into \( 3^{2k+1} \) congruent \((2k + 1)\)-cubes, and let \( A_2 \) be the union of the cubes adjacent to the \( k \)-skeleton of \([0, 1]^{2k+1} \). Repeat the construction on each of the remaining cubes to obtain \( A_3 \) and, similarly, \( A_4, A_5, \ldots \). Then set \( \mu^k = \bigcap_{i=1}^{\infty} A_i \).

**THEOREM.** If \( X \) is a \( k \)-dimensional \((k - 1)\)-connected \((C^{k-1})\), locally \((k - 1)\)-connected \((LC^{k-1})\) compact metric space that satisfies \( DD^k P \), then \( X \approx \mu^k \).

**COROLLARY.** Different constructions of the universal \( k \)-dimensional space appearing in the literature \((cf. [Mg, Lf, Pa])\) yield the same space.

In the proof, a different construction of \( \mu^k \) is used as a working definition. This construction is more suitable for inductive arguments, since it allows “handlebody decompositions” of \( \mu^k \), where each “handle” is a copy of \( \mu^k \), the intersection of two “handles” is a copy of \( \mu^{k-1} \), the intersection of three “handles” is a copy of \( \mu^{k-2} \), etc. This approach leads to a construction of many homeomorphisms \( h: \mu^k \to \mu^k \) which are used to develop a decomposition theory for \( \mu^k \) (via Bing’s Shrinking Criterion). The main result here is that a \( UV^{k-1} \)-surjection \( f: \mu^k \to X \) is approximable by homeomorphisms provided \( \dim X = k \) and \( X \) satisfies \( DD^k P \).

The final part of the proof consists of showing that any \( C^{k-1} \), \( LC^{k-1} \) metric compactum admits a \( UV^{k-1} \)-surjection \( f: \mu^k \to X \) (a resolving map).
Omitting compactness and global connectivity, we obtain the characteriza-
tion of manifolds modeled on $\mu^k$.

**Theorem.** For a locally compact, locally $(k-1)$-connected $k$-dimensional
metric space $X$, the following statements are equivalent.
(i) $X$ satisfies $DD^k P$.
(ii) Each point $x \in X$ admits a neighborhood $U$ homeomorphic to an open
subset of $\mu^k$.

Many theorems from $Q$-manifold theory translate immediately into $\mu^k$-
manifold theory. For example, the $Z$-set unknotting theorem (properly inter-
preted) holds for $\mu^k$-manifolds. In particular, $\mu_k$ is homogeneous (this fact is
well known for $k=0$, and it was proved by R. D. Anderson [An$_1$] for $k=1$).
In short, $\mu_k$ is "the $k$-dimensional analogue of the Hilbert cube $Q$".

**Bibliography**

[Qu] F. Quinn, Resolutions of homology manifolds, and the topological characterization of mani-