LOCAL MODULI
FOR MEROMORPHIC DIFFERENTIAL EQUATIONS

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1. Introduction. This note announces results concerning the parametrization, in the sense of (local) moduli, of the equivalence classes of systems of meromorphic differential equations of the form

(*) \[ \frac{du}{dz} = Au \]

near an irregular singular point (assumed to be \( z = 0 \)). Here \( u \) is an \( n \)-component column vector, \( A \) is an \( n \times n \) matrix of meromorphic functions, and equivalence of systems defined by matrices \( A \) and \( B \) means that there is a meromorphic invertible \( n \times n \) matrix \( x \) such that

(\(\ast\)) \[ x[A] \overset{\text{def}}{=} xAx^{-1} + (dx/dz)x^{-1} = B \]

near \( z = 0 \). If \( \mathcal{F}_{\text{cgt}} \) (resp. \( \mathcal{F} \)) is the field of quotients of the ring of convergent (resp. formal) power series in \( z \) with coefficients in \( \mathbb{C} \), (\(\ast\)) defines an action of \( \text{GL}(n, \mathcal{F}_{\text{cgt}}) \) on \( \text{gl}(n, \mathcal{F}_{\text{cgt}}) \), reflecting the fact that (*) goes over to the system

\[ \frac{dv}{dz} = Bv \]

under the substitution \( v = xu \); replacing \( \mathcal{F}_{\text{cgt}} \) by \( \mathcal{F} \) leads to the notion of formal equivalence. We note that for any commutative ring \( R \) (with unit) equipped with a derivation \( D \), (\(\ast\)) defines an action of \( \text{GL}(n, R) \) on \( \text{gl}(n, R) \), with \( D \) replacing \( d/dz \); if \( R \) is a suitably restricted ring of Laurent series in \( z \) with coefficients in the ring of convergent power series in \( d \) variables and \( D = d/dz \), we obtain the notion of equivalence of analytic families of systems (*) depending on \( d \) parameters, which is basic to the theory of local moduli (cf. [BV2]).

One parametrizes the equivalence classes of systems (*) in two steps. The first step is the classification up to formal equivalence, i.e., the description of the orbit space \( \text{GL}(n, \mathcal{F}) \backslash \text{gl}(n, \mathcal{F}) \); the second step is to fix a formal class \( \Omega \) with \( \Omega_{\text{cgt}} \overset{\text{def}}{=} \Omega \cap \text{gl}(n, \mathcal{F}_{\text{cgt}}) \neq \emptyset \), and to classify the systems (*) in \( \Omega_{\text{cgt}} \) up to equivalence, i.e., to describe the orbit space \( \text{GL}(n, \mathcal{F}_{\text{cgt}}) \backslash \Omega_{\text{cgt}} \). The description of \( \text{GL}(n, \mathcal{F}) \backslash \text{gl}(n, \mathcal{F}) \); goes back to Hukuhara and Turrittin (see [BV1] for extensive references) and is based on the notion of a canonical form. The classical method of studying the second question is based on the technique of Stokes lines and Stokes multipliers [Bi, J]. Recently this has been examined from a more modern, and essentially cohomological, point of view, notably by Malgrange [Ma1, Ma2], Sibuya [S], and Deligne (cf. [Be]). The present note continues this theme by studying the equivalence of analytic families of systems (*) and is based in a fundamental way on the theory of formal equivalence over general rings developed in [BV2].

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For our purposes we define a canonical form to be an element of $\mathfrak{gl}(n, \mathcal{F}_{cgt})$ of the type

$$B = D_{r_1} z^{r_1} + \cdots + D_{r_m} z^{r_m} + z^{-1} C,$$

where (a) $r_1 < r_2 < \cdots < r_m < -1$, the $r_i$ being integers, (b) $C, D_{r_1}, \ldots, D_{r_m}$ are elements of $\mathfrak{gl}(n, \mathbb{C})$ that commute with each other, (c) the $D_{r_i}$ are nonzero and semisimple, and (d) the real parts of all the eigenvalues of $C$ are in $[0, 1)$. For $\Omega$ we take the $\text{GL}(n, \mathbb{F})$-orbit of $B$. We put $\Omega(B) = \Omega_{cgt}$ and write $X(B)$ for the space $\text{GL}(n, \mathcal{F}_{cgt}) \setminus \Omega(B)$. Our main results show (cf. §3) that $X(B)$ may be viewed in a natural way as a space of the form $\mathcal{G}_B \setminus H(B)$, where $H(B)$ is an algebraic variety isomorphic to an affine space $\mathbb{C}^d$ and $G_B$ is an algebraic subgroup of $\text{GL}(n, \mathbb{C})$ acting morphically on $H(B)$, and that “local moduli” exist at the “good” points of this quotient space: the restriction to “good” points is essential even in the simplest cases. Our results may thus be viewed as a description of the analytic deformations of the meromorphic differential equations $du/dz = Au$ when one fixes all the formal invariants of the equation, at least when the point of $H(B)$ defined by $A$ is “smooth and stable”.

2. The Stokes sheaf $\mathcal{S}_B$ and the identification $\text{GL}(n, \mathcal{F}_{cgt}) \setminus \Omega(B) \approx G_B \setminus H^1(\mathcal{S}_B)$. Fix $B$ as in §1 and let $\Psi = \exp \{ \sum_{1 \leq j \leq m} (r_i + 1)^{-1} D_{r_j} z^{r_j+1} \}$. The Stokes sheaf $\mathcal{S}_B$ is the sheaf of (in general noncommutative) groups defined on the unit circle $T$ as follows: for any open subset $U$ of $T$, $\mathcal{S}_B(U)$ is the group of holomorphic maps of the sector $T(U) = \{ z \in \mathbb{C} \mid z \neq 0 \}$ into $\text{GL}(n, \mathbb{C})$ such that $dg/dz = z^{-1} [C, g]$ on $T(U)$. Here, the notation $\sim 1 (\Gamma(U))$ in (a) means that, for any closed arc $U' \subset U$ and any $r \geq 1$, we have $\Psi g \Psi^{-1} - 1 = O(|z|^r)$ as $z \to 0$ in $\Gamma(U')$, the $O$ being uniform in $\Gamma(U')$. If $U$ is an arc $U \subset T$ and $z_U^C = \exp(\log_U z \cdot C)$, where $\log_U$ is a branch of the logarithm on $\Gamma(U)$, the map $g \to z_U^{-C} g z_U^C$ takes $\mathcal{S}_B(U)$ onto a unipotent algebraic subgroup of $\text{GL}(n, \mathbb{C})$ which is independent of the choice of the logarithm. So all the $\mathcal{S}_B(U)$ become unipotent algebraic groups in a natural way. Consequently, if $\mathcal{U} = (U_i)$ is a finite open covering of $T$ by arcs $T$, the set $C(\mathcal{U} : \mathcal{S}_B) = \prod_i \mathcal{S}_B(U_i)$ becomes a unipotent algebraic group, the set $Z^1(\mathcal{U} : \mathcal{S}_B)$ of Čech 1-cocycles becomes an affine variety on which $C(\mathcal{U} : \mathcal{S}_B)$ acts, and the space of orbits can be naturally identified with $H^1(\mathcal{U} : \mathcal{S}_B)$. As usual, $H^1(\mathcal{S}_B)$ is the union of all the $H^1(\mathcal{U} : \mathcal{S}_B)$ as $\mathcal{U}$ varies over the coverings as above. If $G_B$ is the centralizer of $C, D_{r_1}, D_{r_2}, \ldots, D_{r_m}$ in $\text{GL}(n, \mathbb{C})$, $G_B$ acts on each $\mathcal{S}_B(U)$ by $g \to g[u] = g u g^{-1}$, and hence on $H^1(\mathcal{S}_B)$. Our starting point is the following variant of a theorem of Sibuya-Malgrange ([S, Mal]; cf. also [Maj]).

**Proposition 1.** There is a natural map $\theta$ from $\Omega(B)$ to $G_B \setminus H^1(\mathcal{S}_B)$ that is constant on the orbits of $\text{GL}(n, \mathcal{F}_{cgt})$ in $\Omega(B)$ and induces a bijection of $X(B)$ with $G_B \setminus H^1(\mathcal{S}_B)$.

3. The main theorems. By an analytic family $a$ in $\mathcal{F}_{cgt}$ we mean a family $\{ a(\lambda) \} (\lambda \in \Delta^q)$, where $\Delta^q$ is a polydisc in $\mathbb{C}^q$ centered at the origin,
a(\lambda) \in F_{\text{cgt}}$ for all $\lambda \in \Delta^q$, and there is an integer $r \geq 1$ such that, for some holomorphic function $a'$ on $\Delta^q \times \{z \mid |z| < \varepsilon\}$, $a(\lambda)$ is the element of $F_{\text{cgt}}$ defined by $z^{-r}a'(\lambda; z)$. This leads in an obvious way to the notion of analytic families in $gl((n, F_{\text{cgt}})$ and in $GL(n, F_{\text{cgt}})$. If $A$ and $A_1$ are analytic families in $gl((n, F_{\text{cgt}})$ defined over $\Delta^q$, they are called equivalent if there is an analytic family $x$ in $GL(n, F_{\text{cgt}})$ such that $x(\lambda)|A(\lambda)| = A_1(\lambda)$ for all $\lambda$ in some neighbourhood of the origin. An analytic family $A$ in $gl((n, F_{\text{cgt}})$ is said to be in $\Omega(B)$ if $A(\lambda)$ is in $\Omega(B)$ for all $\lambda$ in some neighbourhood of the origin.

Let $\Sigma$ be the set of Laurent polynomials $\sigma = \sum_{1 \leq j \leq m} a_j z^{\sigma_j}$, where $a_j$ is any eigenvalue of $D_{r_j}$, $1 \leq j \leq m$. For $\sigma, \tau \in \Sigma$ with $\sigma \neq \tau$, let $q = q(\sigma, \tau) \leq -2$ be the order of $\sigma - \tau$, $c_q$ the coefficient of $z^q$ in $\sigma - \tau$, and let $S(\sigma, \tau)$ be the (finite) set of rays in $\mathbb{C}^\times$ where $\text{Re}(c_q z^q)$ vanishes. The rays belonging to $\bigcup_{\sigma, \tau \in \Sigma, \sigma \neq \tau} S(\sigma, \tau)$ are called the Stokes lines of $B$. Let $\mathcal{T}(B)$ denote the collection of all finite coverings $U = (U_i)$ of $T$ by open arcs of length $\leq \pi/(|r_1| - 1)$ with the restriction that the ends of the arcs of length equal to $\pi/(|r_1| - 1)$ are not on any Stokes line.

**Theorem 1.** (i) $H^1(\mathcal{S}t_B)$ can be given the structure of an algebraic variety which is natural in the following sense: for any $\mathcal{U} \in \mathcal{T}(B)$, $C(\mathcal{U} : \mathcal{S}t_B)$ acts freely on $Z^1(\mathcal{U} : \mathcal{S}t_B)$, $H^1(\mathcal{U} : \mathcal{S}t_B) = H^1(B)$, and $H^1(\mathcal{S}t_B)$ is the geometric quotient of $Z^1(\mathcal{U} : \mathcal{S}t_B)$ for this action (see [MF] for the notion of geometric quotient); moreover, there is a global cross section for this action.

(ii) $H^1(\mathcal{S}t_B)$ is isomorphic to the affine space $\mathbb{C}^d$, where $d$ is the irregularity of $B$ in the sense of Malgrange (cf. [Be, pp. 233, 238]).

(iii) The action of $G_B$ on $H^1(\mathcal{S}t_B)$ is algebraic.

A point $\gamma \in H^1(\mathcal{S}t_B)$ is called $G_B$-smooth if there exists a $G_B$-invariant open set $U$ containing $\gamma$ such that the geometric quotient $G_B \setminus U$ exists in the category of complex analytic manifolds. Let $H^1(\mathcal{S}t_B)^{\text{sm}}$ be the $G_B$-invariant open set of $G_B$-smooth points. Let $Y = G_B \setminus H^1(\mathcal{S}t_B)$, $\pi$ the natural map $H^1(\mathcal{S}t_B) \to Y$, and $Y^{\text{sm}} = \pi(H^1(\mathcal{S}t_B)^{\text{sm}})$; $Y$ is given the quotient topology. The sheaf of $G_B$-invariant analytic functions on $H^1(\mathcal{S}t_B)$ defines a sheaf on $Y$ and converts $Y$ into a ringed space; and $Y^{\text{sm}}$ is the open subset of points around which this ringed space looks like a complex manifold of dimension $r = d - \delta$, where $\delta$ is the maximum dimension of the $G_B$-orbits in $H^1(\mathcal{S}t_B)$.

**Theorem 2.** Fix $\gamma \in H^1(\mathcal{S}t_B)^{\text{sm}}$. Let $A$ be an analytic family of elements in $\Omega(B)$ defined over $\Delta^q$ such that $\theta(A(0)) = \pi(\gamma)$. Then $\mu(A) : \lambda \mapsto \theta(A(\lambda))$ is an analytic map of a neighbourhood of the origin into a neighbourhood of $\pi(\gamma)$. If $A_1$ is another analytic family in $\Omega(B)$ defined over $\Delta^q$ such that $\mu(A) = \mu(A_1)$ in a neighbourhood of the origin, then $A$ and $A_1$ are equivalent.

The proof of this theorem relies heavily on one of the main results of [BV2].

**Theorem 3.** Let $r$ be as defined earlier. Then we can find an analytic family in $\Omega(B)$ defined over $\Delta^r$ such that $\mu(A)$ is an analytic isomorphism of a neighbourhood of the origin in $\Delta^r$ with a neighbourhood of the point $\pi(\gamma)$. Any such family is universal in the following sense. If $A_1$ is any analytic family in $\Omega(B)$ defined over $\Delta^q$ with $\theta(A_1(0)) = \pi(\gamma)$, we can find an analytic map
$\alpha: \Delta^q \to \Delta^r$ (primes denote concentric polydiscs) vanishing at the origin such that the families $A_1$ and $A \circ \alpha$ are equivalent.

If $C$ is semisimple, $G_B$ is reductive, so we are in the paradigm of Mumford [MF]. Let us call a point $\gamma \in H^1(St_B)$ stable if its $G_B$-orbit is closed and has dimension $\delta$, and let $H^1(St_B)^s$ be the set of stable points; it is $G_B$-invariant and Zariski open. The statement that $H^1(St_B)^s \neq \emptyset$ is equivalent to saying that the action of $G_B$ on $H^1(St_B)$ is generically stable (cf. [MF, p. 154]).

**Theorem 4.** Suppose $C$ is semisimple and $H^1(St_B)^s \neq \emptyset$. Then $Y^\ast = G_B \backslash H^1(St_B)^s$ is an irreducible quasi-affine variety of dimension $r$. If $\Gamma$ is the set of points $\gamma$ in $H^1(St_B)^s$ such that $\pi(\gamma)$ is a simple point in $Y^\ast$, then $\Gamma \subset H^1(St_B)^{sm}$, $\Gamma$ is dense in $H^1(St_B)$, and $G_B \backslash \Gamma$ is a complex manifold of dimension $r$.

Already in simple examples such as the Bessel and Whittaker equations, nonsmooth and smooth nonstable points exist. In general, $G_B \backslash H^1(St_B)^{sm}$ will not be separated. When $B$ is such that the restriction of $C$ to each spectral subspace of $(D_{r_1}, \ldots, D_{r_m})$ has a simple spectrum, then stable points exist, $PG_B = G_B/C^\times$ acts generically freely on $H^1(St_B)$, and $r = d - n + 1$.

**References**


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