Let $M$ be a smooth closed manifold. If $D$ is an elliptic differential operator on $M$, then the differential structure on $M$ is explicitly involved in the definition of the analytic index of $D$. It is a consequence of the Atiyah-Singer Index Theorem that this integer only depends on the homeomorphism type of the manifold $M$, since the topological formula for the index involves the rational Pontrjagin classes which are topological invariants.

By considering families of operators one may determine a more refined index for an elliptic operator which will lie in $K_0(M)$ [1]. This raises the possibility of torsion (i.e., finite order) invariants for operators. We exploit this to study the dependence of the algebra of 0th-order pseudodifferential operators on the underlying differential structure.

The BDF theory of $C^*$-algebra extensions [2] provides a formalism for studying such questions. Recall that the algebra of 0th-order pseudodifferential operators on a manifold $P_0$ defines an extension of $C^*$-algebras $0 \rightarrow K \rightarrow P_0 \rightarrow C(SM) \rightarrow 0$, where $SM$ is the tangent sphere bundle of $M$. We denote this by $P_M \in \text{Ext}(SM)$. There is a natural isomorphism $\Gamma: \text{Ext}(SM) \rightarrow K_1(SM)$. Since $SM$ is a Spin$^c$ manifold, there is a topologically defined $K$-theory fundamental class $[SM] \in K_1(SM)$.

**Theorem 1.** The map $\Gamma: \text{Ext}(SM) \rightarrow K_1(SM)$ satisfies $\Gamma(P_M) = [SM]$.

This follows from the index theorem for families of operators [5].

We now study the question of whether $P_M$ depends on the smooth structure on $M$. Recall that the isotopy classes of smooth structures on $M$ can be made into a finite abelian group $S(M)$. We denote by $M_\alpha$ the manifold $M$ with the differential structure $\alpha \in S(M)$. The identity map $1: M_\alpha \rightarrow M$ induces a map $1: SM_\alpha \rightarrow SM$. There is a unit, $u \in K^0(SM)$, such that $1_*([SM_\alpha]) = u \cap [SM]$. Further, there is a unit $\theta(\alpha) \in K^0(M)$, depending only on the class of $\alpha \in S(M)$, which is a lift of $u$ in the sense that $\pi^*(\theta(\alpha)) = u$, where $\pi: SM \rightarrow M$ is the projection.

Thus, $\theta$ defines a map from $S(M)$ to $K^0(M)$.

**Theorem 2** [5]. The function $\theta: S(M) \rightarrow K^0(M)$ is a homomorphism of $S(M)$ into the multiplicative group of units $1 \oplus \tilde{K}^0(M) \subseteq K^0(M)$.

The next step is to interpret $\theta$ homotopy theoretically. Here one must work separately on the 2-primary and odd-primary parts of $S(M) = S(M)_{(2)} \oplus S(M)_{(\text{odd})}$. The two analyses proceed in a parallel way, so we sketch only
the 2-primary case. (In [5] the odd-primary case was handled by a different method.)

Note first that $S(M) \cong [M, \text{Top}/O]$. A map $\alpha : M \to \text{Top}/O$ can be interpreted as a vector bundle $E$ along with a topological trivialization. Composing $\alpha$ with the natural map into $G/O$ followed by the complexification of Sullivan's map $e : G/O \to BO^\otimes$ yields a unit comparing two orientations of $E$. This defines a homomorphism $e_\mathbb{C} : S(M) \to K^0(M)$ mapping into the multiplicative group of units of $K^0(M)$.

**Theorem 3** [6]. Let $\alpha \in S(M)$.

(i) If $\alpha \in S(M)_{(\text{odd})}$, then $\theta(\alpha) = e_\mathbb{C}(\alpha)^2$.

(ii) If $\alpha \in S(M)_{(2)}$ and, moreover, $M$ is 2-connected, then $\theta(\alpha) = e_\mathbb{C}(\alpha)^2$.

It follows from (i) and the odd-primary analysis of the fibration

$$\text{Top}/O \xrightarrow{i} G/O \xrightarrow{j} G/\text{Top}$$

due to Sullivan [9] that we have

**Theorem 4.** If $\alpha \in S(M)_{(\text{odd})}$, then $\theta(\alpha) = 1$.

The 2-primary case is different. Here, we use the analysis of (1) localized at 2 due to Brumfiel, Madsen and Milgram [3]. We construct a finite complex $X$ and a map $\alpha : X \to \text{Top}/O$ for which $(e_\mathbb{C} \circ \alpha)$ is not null-homotopic. By embedding $X$ in a sphere and taking the double of a smooth regular neighborhood, one obtains a smooth manifold $M$. Using this manifold and the smooth structure determined by the map $\rho \alpha$, where $\rho$ is a retraction of $M$ onto $X$, we obtain the following theorem.

**Theorem 5** [6]. There is a smooth manifold $M$ with a second differential structure $\alpha \in S(M)_{(2)}$, for which $\theta(\alpha) \neq 1$.

Thus $P_M$ can, indeed, depend on the smooth structure.

**Corollary 6.** The algebra of 0th-order pseudodifferential operators on $M$ depends on the differential structure.

Our construction yields an infinite family of such manifolds. However, one may also construct manifolds $M$ and smooth structures in the 2-primary part of $S(M)$ for which the invariant $\theta(\alpha)$ is trivial.

These results can be interpreted in the following way. Let $M$ be a smooth closed manifold. There is a Poincaré duality map in $K$-theory:

$$K^0(TM) = K^0(DM, SM) \to K_0(DM) = K_0(M).$$

If one uses Atiyah's version of $K_0(M)$ [1], this map sends the symbol of an operator to the class of the operator considered as an element of $K_0(M)$. It follows from Theorem 5 that this Poincaré duality map depends on the differential structure.

These notions have been set in the framework of families of operators by A. Connes and G. Skandalis [4] in their work on index theory for foliated manifolds. They define a map $\psi^* : K^0(TM \times X) \to KK(M, X)$, which may be viewed as sending the symbol of a family of operators on $M$, parametrized by...
the compact space $X$, to the element of the Kasparov group [7] defined by that family. Again Theorem 5 implies that $\psi^*$ depends on the differential structure on $M$. In this sense the index theorem for families is not topologically invariant, as opposed to the ordinary index theorem for a single operator.

REFERENCES