1. Arnold's conjecture. An automorphism \( \psi \) of a symplectic manifold \((P, \omega)\) is homologous to the identity if there is a smooth family \( \psi_t \) \((t \in [0, 1])\) of automorphisms such that the time-dependent vector field \( \xi_t \) defined by \( d\psi_t/dt = \xi_t \circ \psi_t \) is globally hamiltonian; i.e. if there is a smooth family \( H_t \) of real-valued functions on \( P \) such that \( \xi_t \circ \omega = dH_t \). It was conjectured by Arnold [1], as an extension of the Poincaré-Birkhoff annulus theorem [3, 7], that every automorphism of a compact symplectic manifold \( P \), homologous to the identity, has at least as many fixed points as a function on \( P \) has critical points.

Arnold’s conjecture was proven by Conley and Zehnder [4] for the torus \( T^{2n} \approx \mathbb{R}^{2n}/\mathbb{Z}^{2n} \) with its usual symplectic structure. They show that every symplectic automorphism of \( T^{2n} \), homologous to the identity, has at least \( n + 1 \) fixed points, and at least \( 2^{2n} \) if all are nondegenerate. Their method was extended in [8] to prove a version of Arnold’s conjecture for arbitrary \( P \) under the additional assumption that the hamiltonian vector field \( \xi_t \) is sufficiently \( C^0 \) small.

In this note we announce a proof of Arnold’s conjecture for the complex projective space \( \mathbb{CP}^n \) with its standard symplectic structure. We prove that a symplectic diffeomorphism of \( \mathbb{CP}^n \), homologous to the identity, has at least \( n + 1 \) distinct fixed points. (By the Lefschetz fixed point theorem, any continuous map from \( \mathbb{CP}^n \) to itself, homotopic to the identity, has at least \( n + 1 \) fixed points counted with multiplicities.) For \( n = 1 \) (\( \mathbb{CP}^1 \approx S^2 \)) the result was already known [1], but with a proof which worked only in this two-dimensional case.

The proof for \( T^{2n} \) in [4] made use of a variational principle in which the fixed points of the map were identified with periodic solutions of a time-dependent hamiltonian system and then identified with critical points of a functional on the space of contractible loops on \( T^{2n} \). The corresponding functional in the case of \( \mathbb{CP}^n \) is multiple valued, and there are other difficulties connected with the curved geometry of \( \mathbb{CP}^n \), so we need a new approach. Our trick is to consider the hamiltonian system on \( \mathbb{CP}^n \) as the reduction, in the sense of [6], of a hamiltonian system on \( \mathbb{C}^{n+1} \) and then adapt recently developed methods [2] for finding periodic orbits in \( \mathbb{C}^{n+1} \). This method is similar to that of Conley and Zehnder in that a problem on a compact manifold is lifted to a problem on euclidean space invariant under a group of transformations.

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2. Lifting to $\mathbb{C}^{n+1}$. Consider $\mathbb{C}^{n+1}$ with its usual symplectic structure $\text{Im} \sum dz_i \wedge d\bar{z}_i$. The Hamiltonian $K(z) = \sum z_i \bar{z}_i$ generates the periodic flow $T_\mu(z_1, \ldots, z_{n+1}) = (e^{2\mu z_1}, \ldots, e^{2\mu z_n})$ with period $\pi$, and hence an action of $S^1 = \mathbb{R}/\pi \mathbb{Z}$ (the Hopf fibration). The reduced manifold $K^{-1}(1)/S^1$ can be identified with $\mathbb{C}P^n$, and any $S^1$-invariant Hamiltonian system on $\mathbb{C}^{n+1}$ induces a system on $\mathbb{C}P^n$, called the reduced system. Our idea is to use this procedure in the opposite direction.

Fixed points of $\psi : \mathbb{C}P^n \to \mathbb{C}P^n$ are the same as solution curves $\tilde{\sigma} : [0,1] \to \mathbb{C}P^n$ with $\tilde{\sigma}(0) = \tilde{\sigma}(1)$ for the time-dependent Hamiltonian system which generates the family $\psi_t$ connecting the identity to $\psi$. Let $\tilde{H}_t$ be the Hamiltonian family for this system; since each $\tilde{H}_t$ contains an arbitrary constant, we may assume that $\tilde{H}_t(z) > 0$ for all $t$ in $[0,1]$ and all $z$ in $\mathbb{C}P^n$. Now let $H_\mu : \mathbb{C}^{n+1} \to \mathbb{R}$ be the unique function which is homogeneous of degree 2 and whose restriction to $K^{-1}(\mu) = S^{2n+1}$ is the pullback of $\tilde{H}_t$. Then $H_\mu$ is $S^1$-invariant and defines a time-dependent Hamiltonian system on $\mathbb{C}^{n+1}$ whose reduced system is $H_t$.

By the general theory of reduction, we know that $S^{2n+1}$ is an invariant manifold for $H_\mu$, and the orbits of $H_\mu$ on $\mathbb{C}P^n$ are the images of orbits of $H_\mu$ on $S^{2n+1}$. Furthermore, if $\tilde{\sigma}$ is the image of $\sigma$, then $\tilde{\sigma}(1) = \tilde{\sigma}(0)$ if and only if $\sigma(1) = T_\mu \sigma(0)$ for some $\mu$ in $\mathbb{R}/\pi \mathbb{Z}$. If we change the Hamiltonian $H_\mu$ to $H_\mu + \lambda K$ for some $\lambda \in \mathbb{R}$, then the “flow” of $H_\mu + \lambda K$ will still project to that of $\tilde{H}_t$, but now by choosing $\lambda (\text{mod} \pi) = \mu$ we can make $\sigma(1) = \sigma(0)$. In other words, to each closed solution curve $\tilde{\sigma}$ for $\tilde{H}_t$ and, hence, to each fixed point of $\psi$ there corresponds a collection of pairs $(\sigma, \lambda)$ where $\lambda \in \mathbb{R}$ and $\sigma$ is a closed solution curve for $H_\mu + \lambda K$ on $S^{2n+1}$. The set of all pairs $(\sigma, \lambda)$ corresponding to a given fixed point is diffeomorphic to $S^1 \times \mathbb{Z}$.

By Hamilton’s principle the closed solution curves for $H_\mu + \lambda K$ on $\mathbb{C}^{n+1}$ are exactly the critical points of the functional

$$g(z) = \int_0^1 -i(z'(t), z(t)) \, dt + \int_0^1 H_\mu(z(t)) \, dt + \lambda \int_0^1 |z(t)|^2 \, dt$$

$$= A(z) + H(z) + \lambda K(z).$$

Since we are interested in critical points for all possible values of $\lambda$, we may consider $\lambda$ as a Lagrange multiplier and look for critical points of $f(z) = A(z) + H(z)$ constrained to the infinite-dimensional sphere $K^{-1}(1)$.

We are thus faced with two problems. The first is to do the analysis which shows that $f(z)$ has many critical points on $K^{-1}(1)$, and the second is to show that all these critical points cannot belong to fewer than $n+1$ families of type $S^1 \times \mathbb{Z}$ coming from distinct fixed points of $\psi$.

3. Critical point analysis. The solution of the problems stated at the end of §2 forms the content of [5] and will only be summarized briefly here.

It turns out that the critical point theory developed in [2], based on the notion of relative index, is applicable to our problem, with some modifications made to permit working on the sphere $K^{-1}(1)$ within the space of loops of Sobolev class $H^{1/2}$ in $\mathbb{C}^{n+1}$. The values of the Lagrange multiplier $\lambda$ are then found to be equal to the critical values of the functional $f$ on $K^{-1}(1)$. 

The minimax nature of the critical point theory makes it possible to estimate these values by comparison with the action functional $A$. A combinatorial argument then shows that these critical values cannot lie in less than $n + 1$ cosets of $\mathbf{R}$ (mod $\pi \mathbf{Z}$) unless some critical values merge, in which case $\psi$ would have uncountably many fixed points.

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