Finiteness of Mordell–Weil Groups of Generic Abelian Varieties

By Alice Silverberg

In a series of papers in the 1960s Shimura studied analytic families of abelian varieties with fixed polarization, endomorphism, and level structure. The isomorphism classes of abelian varieties in such a family are in one-to-one correspondence with the points of $D/\Gamma$, where $D$ is a symmetric domain and $\Gamma$ is a discontinuous group of transformations of $D$. Shimura constructed a fibre system $(V, W)$ where the base $V$ is analytically isomorphic to $D/\Gamma$, the fibres are the abelian varieties in the family, and $V$ and $W$ are quasi-projective varieties. The fibre $A$ over the generic point of $V$ is an abelian variety defined over the function field $K$ of $V$. The main result of this announcement is that, under certain conditions on the endomorphism algebra structure, the group of points of $A$ defined over $K$ is finite. Using completely different techniques, Shioda [8] proved this result in the case in which $D$ is the complex upper half-plane and $\Gamma$ is a congruence subgroup of $\text{SL}_2(\mathbb{Z})$.

The results in this note are an extension of part of the author’s Ph.D. thesis [9]. Details will appear elsewhere. I would like to express my sincere thanks to my thesis advisor, Professor Goro Shimura.

1. Let $F$ be an arbitrary totally real number field of degree $g$ over the rational number field $Q$. Let $L$ be either (a) the field $F$, (b) a totally indefinite quaternion algebra over $F$ (and view $L$ as embedded in $M_2(\mathbb{R})^\theta$), or (c) a totally imaginary quadratic extension $K$ of $F$. Let $\Phi$ be a representation of $L$ by complex matrices of degree $n$ so that $\Phi + \Phi$ is equivalent to a rational representation of $L$, and $\Phi(1) = 1_n$ (writing $1_n$ for the identity matrix of size $n$). Assume that $[L : Q]$ divides $2n$, and let $m = 2n/[L : Q]$. In (c), if $r_1, \ldots, r_g, \bar{r}_1, \ldots, \bar{r}_g$ are the distinct embeddings of $K$ in the complex number field $\mathbb{C}$, write $r_\nu$ and $s_\nu$, respectively, for the multiplicities of $r_\nu$ and $\bar{r}_\nu$ in $\Phi$ (then $r_\nu + s_\nu = m$). Suppose $T \in M_m(L)$ satisfies $^tT\rho = -T$, where $^t$ is transpose on $M_m(L)$, and $\rho$ is complex conjugation on $K$ and transpose on each factor of $M_2(\mathbb{R})^\theta$. In (c), suppose $iT^{r_\nu}$ has the same signature as

$$
\begin{pmatrix}
1_{r_\nu} & 0 \\
0 & -1_{s_\nu}
\end{pmatrix}
$$

for every $\nu$. Let $M$ be a lattice in $L^m$, and let $v_1, \ldots, v_s$ be elements of $L^m$. Let $\Omega$ denote the collection of data $(L, \Phi, \rho, T, M, v_1, \ldots, v_s)$.

Suppose $A$ is an abelian variety with a polarization $\mathcal{C}$, $\theta$ is an embedding of $L$ into $\text{End}(A) \otimes Q$, and $t_1, \ldots, t_s$ are elements of $A$ of finite order.
DEFINITION. \((A, C, \theta, t_1, \ldots, t_s)\) is a polarized abelian variety of type \(\Omega\) if

1. there is a holomorphic mapping \(\xi\) of \(\mathbb{C}^n\) onto \(A\) inducing an isomorphism of a complex torus \(\mathbb{C}^n/Y\) onto \(A\) satisfying \(\xi(\Phi(a)u) = \theta(a)\xi(u)\) for every \(u \in \mathbb{C}^n\) and \(a \in \Theta^{-1}(\text{End}(A))\); 
2. if \(\gamma\) is the involution of \(\text{End}(A) \otimes \mathbb{Q}\) determined by \(C\), then \(\theta(a)\gamma = \theta(a^o)\) for every \(a \in L\); 
3. there is an \(\mathbb{R}\)-linear isomorphism \(\eta\) of \((L \otimes \mathbb{Q} \mathbb{R})^m\) onto \(\mathbb{C}^n\) such that \(\eta(\mathcal{M}) = Y, \ t_i = \xi(\xi(v_i))\) for \(i = 1, \ldots, s\), and \(\eta(ax) = \Phi(a)\eta(x)\) for every \(a \in L\) and \(x \in (L \otimes \mathbb{Q} \mathbb{R})^m\); and
4. \(C\) determines a Riemann form \(R\) on \(\mathbb{C}^n/Y\) such that \(R(\eta(x), \eta(y)) = \text{tr}(xT^y\gamma^o)\) for every \(x\) and \(y\) in \((L \otimes \mathbb{Q} \mathbb{R})^m\).

Write \(H_r\) for \(\{Z \in M_r(\mathbb{C})| \text{tr}(Z) = 0; \text{Im}(Z)\text{ is positive symmetric}\}\) and \(H_{r,s}\) for \(\{Z| \text{complex matrix with } r \text{ rows and } s \text{ columns}; 1 - Z^t Z \text{ is positive}\}\). Let \(L\) be \(H_{m/2}^0\) in (a), \(H_m^0\) in (b), and \(H_{r_1,s_1} \times \cdots \times H_{r_g,s_g}\) in (c).

The isomorphism classes of polarized abelian varieties of type \(\Omega\) are in one-to-one correspondence with the points of \(D/T\), where \(T\) is a suitably defined group of transformations on \(D\) (see [3] and [4]). In [5] Shimura showed that for each \(\Omega\), one can construct a fibre system \(\mathcal{F}\) in which the base \(V\) is analytically isomorphic to \(D/T\) and the fibres are the polarized abelian varieties of type \(\Omega\).

**THEOREM 1.** If \(\dim(V) \geq 1\) then the group of points of the generic fibre defined over the function field of \(V\) is finite.

The remainder of this paper is a sketch of the proof of Theorem 1.

2. The Mordell-Weil group of Theorem 1 is isomorphic to the group of rationally defined algebraic sections from the base \(V\) to the fibre variety \(W\). If \(V\) is one-dimensional, one sees easily that these sections extend to global holomorphic sections. For higher dimensions we have the following result, which is a consequence of a result of Igusa (Theorem 6 of [1]) when the base variety \(V\) is compact.

**PROPOSITION.** Let \(f\) be a rational section from \(V\) to \(W\). Then \(f\) is defined at every point of \(V\) so that \(f\) gives a holomorphic section from \(V\) to \(W\).

When \(\dim(V) = 1\) and \(V\) is compact, the second derivative of a holomorphic section is an automorphic form of weight three with respect to \(\Gamma\). The Eichler-Shimura cohomology isomorphism (Theorem 8.4 of [7]) can be used to show these automorphic forms are zero, and this then restricts the number of holomorphic sections. When \(D\) is \(H_r^0\) or \(H_{r,1}^0\) with \(r > 1\), the use of the Eichler-Shimura cohomology isomorphism is replaced by the application of a theorem of Matsushima and Shimura (Theorem 3.1 of [2]), which says there are no automorphic forms of mixed weight with at least one nonpositive weight.

3. The cases of Theorem 1 discussed in §2 can be used to prove the theorem in the remaining cases. We select a large collection of embeddings of base varieties \(V\), for which the theorem is known, into a variety \(V\) for which we want to prove the theorem. A section \(f\) over \(V\) may be pulled back to sections over the varieties \(V\). Since every section over every \(V\) is of finite order, we can obtain a dense set of points of \(V\) which map via \(f\) to points of finite order.
in the fibres over $V$. To show $f$ is torsion, we must show these orders are bounded. We do this by proving a theorem giving a uniform bound for orders of torsion points on fibres with complex multiplication (Theorem 2 below). The finiteness of the Mordell-Weil group of the generic fibre then follows.

For $u$ in $V$, write $Q_u = (A_u, C_u, \theta_u, t_1(u), \ldots, t_s(u))$ for the fibre over $u$. The fibre system $\mathcal{F}$ is defined over a number field $k_u$ of finite degree such that for every $u \in V$, $k_u(u)$ is the field of moduli of $Q_u$ (see [5]). Call $Q_u$ a "CM-fibre" if $A_u$ is isogenous to $A_1 \times \cdots \times A_t$, where $A_i$ has complex multiplication by a $CM$-field of degree $2 \cdot \dim(A_i)$, for $i = 1, \ldots, t$ (thus, $A_u$ has $CM$ in the sense of [6]).

**Theorem 2.** Let $k$ be any subfield of $\mathbb{C}$ which is finitely generated over $\mathbb{Q}$ and contains $k_u$. There is a constant $B$, depending only on the field $k$ and the fibre system $\mathcal{F}$, and independent of the choice of $CM$-fibre $Q_u$, so that $|A_u(k(u))_{\text{torsion}}| \leq B$.

The proof of Theorem 2 requires Shimura's Main Theorem of Complex Multiplication.

**References**