GEL’FAND’S PROBLEM ON UNITARY REPRESENTATIONS
ASSOCIATED WITH DISCRETE SUBGROUPS OF $\text{PSL}_2(\mathbb{R})$

BY TOSHIKAZU SUNADA

In 1978 M.-F. Vignéras [10] gave a negative answer to the question posed by I. M. Gel’fand [2], who asked if the induced representation of $\text{PSL}_2(\mathbb{R})$ on $L^2(\Gamma \backslash \text{PSL}_2(\mathbb{R}))$ determines a discrete subgroup $\Gamma$ up to conjugation. She constructed explicitly two nonconjugate discrete groups arising from indefinite quaternion algebras defined over number fields giving rise to isomorphic induced representations. Such examples are necessarily quite sporadic since there are only finitely many conjugacy classes of arithmetic groups with a fixed signature; see K. Takeuchi [9]. The purpose of this note is to give, in a rather simple way, a large family of (nonarithmetic) discrete groups that are not determined by their induced representations. The key idea is to reduce the problem to the case of finite groups where the situations are simple and well understood. We should point out that a similar idea can be applied to constructions of various isospectral Riemannian manifolds [8].

Our construction is based on the following proposition, which follows from standard facts about induced representations.

**Proposition.** Let $G$ be a locally compact topological group, and let $\Gamma \subset \Gamma_1, \Gamma_2 \subset \Gamma_0$ be discrete subgroups, with $\Gamma$ normal and of finite index in $\Gamma_0$. Then if the subgroups $\mathcal{G}_i = \Gamma_i/\Gamma, \ i = 1, 2$, of $\mathcal{G} = \Gamma_0/\Gamma$ meet each conjugacy class of $\mathcal{G}$ in the same number of elements, the representations of $G$ on the spaces $L^2(\Gamma \backslash G), \ i = 1, 2$, are unitarily equivalent.

To construct nonconjugate $\Gamma_1$ and $\Gamma_2$ in $\text{PSL}_2(\mathbb{R})$, we first choose an appropriate triple $(\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2)$ with the same induced representation $\text{Ind}_{\mathcal{H}_1}^{\mathcal{G}}(1) \equiv \text{Ind}_{\mathcal{H}_2}^{\mathcal{G}}(1)$. For instance, we let $\mathcal{G}$ be the semidirect product $(\mathbb{Z}/8\mathbb{Z}) \rtimes (\mathbb{Z}/8\mathbb{Z})$ and set $\mathcal{H}_1 = \{(1,0), (3,0), (5,0), (7,0)\}$, $\mathcal{H}_2 = \{(1,0), (3,4), (5,4), (7,0)\}$. It is easy to check that $\mathcal{H}_1$ and $\mathcal{H}_2$ are not conjugate in $\mathcal{G}$, and each conjugacy class of $\mathcal{G}$ meets $\mathcal{H}_1$ and $\mathcal{H}_2$ in the same number of elements.

We then take a torsion-free discrete subgroup $\Gamma_0 \subset \text{PSL}_2(\mathbb{R})$ satisfying the following conditions:

(i) the genus of the Riemann surface $\Gamma_0 \backslash \text{PSL}_2(\mathbb{R})/\text{SO}(2)$ is greater than two;

(ii) $\Gamma_0$ is maximal in $\text{PGL}_2(\mathbb{R})$; and

(iii) $\Gamma_0$ is nonarithmetic.

Since the fundamental group of a Riemann surface of genus $k$ has a free group on $k$ generators as quotient, we may always, if $k$ is large enough, find a sur-
jective homomorphism \( \varphi : \Gamma_0 \to \mathcal{G} \). Since the example of \( \mathcal{G} \) given in the previous paragraph has three generators, for it we may take \( k = 3 \). If we put \( \Gamma_i = \varphi^{-1}(\mathcal{H}_i) \), then, in view of the Proposition, \( \Gamma_1 \) and \( \Gamma_2 \) give rise to isomorphic representations. But \( \Gamma_1 \) and \( \Gamma_2 \) are not conjugate in \( \text{PGL}_2(\mathbb{R}) \). In fact, if \( g \Gamma_1 g^{-1} = \Gamma_2 \) for some \( g \in \text{PGL}_2(\mathbb{R}) \), then \( g \Gamma_0 g^{-1} \) is commensurable with \( \Gamma_0 \). A result announced by G. A. Margulis [6] implies that the commensurability group \( C(\Gamma_0) = \{ h \in \text{PGL}_2(\mathbb{R}) ; h \Gamma_0 h^{-1} \text{ is commensurable with } \Gamma_0 \} \) is discrete provided that \( \Gamma_0 \) is nonarithmetic. Since \( C(\Gamma_0) \supseteq \Gamma_0 \), from the maximality of \( \Gamma_0 \), it follows that \( g \in \Gamma_0 \); hence, \( \varphi(g) \mathcal{H}_1 \varphi(g)^{-1} = \mathcal{H}_2 \). This is a contradiction.

Existence of “many” \( \Gamma_0 \) satisfying (i)–(iii) is guaranteed by results of L. Greenberg [4], A. M. Macbeath and D. Singerman [5], and K. Takeuchi [9]. In fact, by [4], generic \( \Gamma_0 \) are maximal in \( \text{PSL}_2(\mathbb{R}) \). If such a \( \Gamma_0 \) is not maximal in \( \text{PGL}_2(\mathbb{R}) \), then the normalizer \( N(\Gamma_0) \) of \( \Gamma_0 \) in \( \text{PGL}_2(\mathbb{R}) \) is strictly bigger than \( \Gamma_0 \), so that the isometry group \( N(\Gamma_0)/\Gamma_0 \) of the surface \( \Gamma_0/\text{PSL}_2(\mathbb{R})/\text{SO}(2) \) is not trivial. On the other hand, by [5], the isometry group is trivial for generic \( \Gamma_0 \). Combining these facts with finiteness of arithmetic groups [9], we get the genericity of \( \Gamma_0 \) satisfying (i)–(iii).

**Remark.** (a) For the above groups \( \Gamma_i \) the genus of the surface \( \Gamma_i/\text{PSL}_2(\mathbb{R})/\text{SO}(2) \) equals \( 8k - 7 \) (\( k \geq 3 \)). The examples given by Vignéras have much bigger genus.

(b) Examples of \( (\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2) \) satisfying the condition in the Proposition have been used by number-theorists to construct nonisomorphic number fields with the same Dedekind zeta function (see, for instance, [7]). It is also known that many examples of the triple \( (\mathcal{G}, \mathcal{H}_1, \mathcal{H}_2) \) arise from simple algebraic groups: If \( \mathcal{G} \) is a reductive algebraic group and \( \mathcal{H}_1, \mathcal{H}_2 \) are nonconjugate parabolic subgroups, then \( \text{Ind}_{\mathcal{G}}^{\mathcal{G}}(1) \), \( i = 1, 2 \), are isomorphic.

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**References**

7. R. Perlis, *On the equation \( \zeta_k(s) = \zeta_l(s) \)*, J. Number Theory **9** (1977), 342–260.

Department of Mathematics, Nagoya University, Nagoya 464, Japan