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A GENERALIZATION OF BERGER'S ALMOST $\frac{1}{4}$-PINCHED MANIFOLDS THEOREM. I

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1. Introduction. Let M^n, $K(M)$, $d(M)$, $i(M)$ denote a compact smooth Riemannian manifold of dimension n, its sectional curvature, diameter, and injectivity radius, respectively. The classical Sphere Theorem states that any simply connected Riemannian manifold M, with $1 \leq K(M) < 4$, is homeomorphic to a sphere [**1**; **5**, p. 107; **17**]. A generalization of that is given by Grove and Shiohama [**14**]: If $K(M) \geq 1$ and $d(M) > \pi/2$, then M is homeomorphic to a sphere. $\mathbf{C}P^n$, $\mathbf{H}P^n$, $\mathrm{Ca}\,P^2$ and $\mathbf{R}P^n$, with their standard metrics, have $1 \leq K(\cdot) \leq 4$ and $d(\cdot) = \pi/2$. Berger also showed that any simply connected Riemannian manifold with $1 \leq K \leq 4$ is either homeomorphic to a sphere or isometric to a compact symmetric space of rank one—i.e., either $\mathbf{C}P^n$, $\mathbf{H}P^n$, or $\mathrm{Ca}\,P^2$ [**1**]. The generalization of this result for the diameter case is given by Gromoll and Grove [**9**]: If $K(M) \geq 1$ and $d(M) \geq \pi/2$, then M either is homeomorphic to a sphere, isometric to $\mathbf{C}P^n$, $\mathbf{H}P^n$, or has the cohomology ring of $\mathrm{Ca}\,P^2$, when it is simply connected. In [**9**] the nonsimply connected case also is classified. Recently, Berger [**2**] showed that there exists $\varepsilon > 0$, depending on the dimension, such that any simply connected Riemannian manifold M^n with $1 \leq K(M) \leq 4 + \varepsilon$, n even, is either homeomorphic to a sphere or diffeomorphic to $\mathbf{C}P^n$, $\mathbf{H}P^n$, or $\mathrm{Ca}\,P^2$.

The main purpose of this paper is to announce some generalizations of [**2**], which extends [**9**]. The author was informed that S. Peters has obtained some results in the direction of Theorem 3. Theorem 3 is a necessary tool in our proof, but it is not the main aim. The author wishes to thank D. Gromoll, K. Grove, and W. Meyer for helpful discussions and for bringing to his attention the fact that the limit metric is possibly C^1.

2. Main results.

THEOREM 1. *Let $n \geq 2$, $K \geq 4$, $\delta > 0$. There exists $\varepsilon = \varepsilon(K, n, \delta) > 0$ such that, for any n-dimensional smooth Riemannian manifold M with*
(i) $1 \leq K(M) \leq K$,
(ii) $d(M) > \pi/2 - \varepsilon$, *and*
(iii) $i(M) > \delta$, *if n is odd,*
we have either
(a) *M is homeomorphic to a sphere, or*
(b) *$\pi_1(M) = 0$ and $H^*(M, \mathbf{Z})$ is a truncated polynomial ring with one generator in $H^\lambda(M, \mathbf{Z})$, where $n = k\lambda$, n is even, $k \in \mathbf{N}$, $k \geq 2$, $\lambda = 2, 4$, or 8, and if $\lambda = 8$ then $k = 2$ and $n = 16$, or*

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(c) $\pi_1(M) \neq 0$ *and there exists a* C^∞ *Riemannian metric* g' *on* M *with* $K(M,g') \geq 1$ *and* $d(M,g') = \pi/2$; *that is, the universal cover* $(\tilde{M},\tilde{g}')$ *of* (M,g') *is isometric to* $S^n(1)$ *or* $\mathbf{C}P^{\mathrm{odd}}$.

REMARKS. (1) If n is even, then (iii) is irrelevant and $\varepsilon = \varepsilon(K,n)$ since $i(M) \geq \pi/2\sqrt{K}$.

(2) Cases (a) and (c) are the best possible conclusions. In (b) the best conclusion would be "diffeomorphic to $\mathbf{C}P^k$, $\mathbf{H}P^k$, or $\mathrm{Ca}\,P^2$". We have the following extension of (b) with stronger hypothesis and conclusion. Theorems 1 and 2 together generalize [**2**], since, in even dimensions, $1 \leq K \leq 4+\varepsilon$ implies that $i(M) \geq \pi/\sqrt{4+\varepsilon}$ for simply connected M, and the hypothesis for $k \geq 3$ is weaker than a bound on $i(M)$.

THEOREM 2. *Let* M *satisfy the hypotheses of Theorem* 1, *conclusion* (b), *and either*

(i) $k \geq 3$ *and there exist* $p_1, p_2, p_3 \in M$ *with* $d(p_i,p_j) = (1-\delta_{ij})\,d_M$, *or*

(ii) $k = 2$ *and* $i(M) \geq \pi/2 - \varepsilon$.

Then M *is diffeomorphic to* $\mathbf{C}P^k$, $\mathbf{H}P^k$, *or* $\mathrm{Ca}\,P^2$.

3. A sketch of the proofs. The main idea is to use Gromov's Compactness Theorem [**12**, p. 129] and give a similar proof to [**9**] on a limit metric which may not be smooth; then, using the finiteness theorems [**4, 5, 18**], we conclude the existence of such ε's. This is similar to the ideas used in [**2**], which is modelled on [**1**] instead of [**9**].

Step 1. The number of diffeomorphism types of manifolds M satisfying $1 \leq K(M) \leq K$, $i(M) > \delta$, $d(M) \leq \pi/2$ for fixed δ and K is finite [**4**]. Suppose there exists a diffeomorphism class $[M]$ which admits C^∞-Riemannian metrics g_n, with $\pi/2 \geq d(M,g_n) \geq \pi/2 - 1/n$, $\forall n \in \mathbf{N}^+$, $i(M,g_n) > \delta$, and $1 \leq K(M,g_n) \leq K$. Using [**12**] we obtain that a subsequence of g_n converges to a Riemannian metric g_0 on M, which is a priori C^0, with respect to Lipschitz or Hausdorff metrics on the space of Riemannian manifolds. Any topological or geometric property of (M,g_0), up to diffeomorphism, will be a property of $[M]$.

Step 2. Even g_0 is C^0; since it is the limit of metrics satisfying Toponogov's Theorem [**5**, p. 43] for $K(\cdot) \geq 1$, one can show that a simple form of Toponogov's Theorem holds for (M,g_0). Using this we can obtain the following

LEMMA. *Let* $p,q \in (M,g_0)$, $v \in U(M,g_0)_p$ *with* $d(p,q) \leq d(q,\exp_p tv) \leq \pi/2$, $\forall t \in [-\delta,\delta]$, $0 < \delta \ll d(q,p)$. *Let* γ *and* θ *be two minimal geodesics from* q *to* p *and* $r := \exp_p \delta v$, *respectively, such that* $0 < \measuredangle\,(\dot\gamma(q),\dot\theta(q)) \ll \pi/2$. *Then* $d(\exp_p \delta w, \exp_p \delta v) < d(q,p)$, *where* w *is the unique vector in* $U(M,g_0)_q$ *such that*

$$\measuredangle\,(w,\dot\theta(q)) + \measuredangle\,(\dot\theta(q),\dot\gamma(q)) = \measuredangle\,(w,\dot\gamma(q)) = \pi/2.$$

REMARK. Such a result can easily be obtained in a C^∞ Riemannian manifold with positive curvature using Rauch II, but g_0 is not necessarily C^2.

Step 3. THEOREM 3. g_0 *is in fact* C^1, *in some differentiable coordinate charts.*

To prove that, we use harmonic coordinates and estimates obtained by Jost and Karcher [**16**]. As in [**2**], the exponential map is defined, C^0, and unique on (M, g_0).

The rest of the proof follows [**9**] very closely in the main steps, but most of the proof has to be modified, especially in Step 5, since the metric is not C^∞.

Step 4. $d(M, g_0) = \pi/2$. There is a pair of dual sets A and B in M such that $A = \{x \in M | d(x, B) = \pi/2\}$ and $B = \{x \in M | d(x, A) = \pi/2\}$. A and B are convex, totally geodesic C^1 submanifolds of M, with or without boundary.

Step 5. By using Step 2: If $\partial A \neq \emptyset$, then A is contractible. In fact, $\exists p_0 \in A$ with $d(p_0, \partial A) = \max\{d(x, \partial A) | x \in A\}$, and all critical points of $d(p_0, \cdot)$, in the sense of Gromov [**13**], lie on B. Consequently, if $\partial A \neq \emptyset$ and $\partial B \neq \emptyset$, then M is homeomorphic to a sphere.

Step 6. If $p, r \in A$, $q \in B$, γ_1 is a minimal geodesic from p to r of length α, and γ_2 is a minimal geodesic from q to p, then there exists a minimal geodesic γ_3 from q to r and a 2-surface L bounded by γ_1, γ_2, and γ_3, and L is totally geodesic and isometric to the interior of a triangle in $S^2(1)$ with side lengths α, $\pi/2$, $\pi/2$. Also, see [**2**, Lemma 8]. Using these 2-surfaces and Step 3, we can define limited forms of parallel transport and holonomy.

Step 7. If $\partial A = \emptyset$, then for any point $p \in B$, there is a fibration of the unit normal sphere UNB_p with fibers given by the link $L(p, q) = \{\dot\gamma(p) | \gamma$ is a normal minimal geodesic from p to $q\}$, $\forall q \in A$, and the quotient space is A. The projection map onto A is explicitly given by the minimal geodesics from p to A. Using the normal holonomy of A and Step 3, $L(p, q)$ is a closed $(\lambda - 1)$-dimensional submanifold of UNB_p, and the local triviality of the fibration can be obtained from the translation defined by using Step 6 (i.e., sending γ_2 to γ_3). So all $L(p, q)$ are homeomorphic for all $q \in A$, and $L(p, q) \hookrightarrow \mathrm{UNB}_p \to A$ is a fiber bundle. By Browder [**3**] each component of $L(p, q)$ has the homotopy type of S^1, S^3, S^7, or a point. Define $a = \dim A$ and $b = \dim B$. We have $a + b + \lambda = n$, $\lambda = 1, 2, 4, 8$. If $\lambda = 8$ then $a, b \leq 8$. $H^*(\tilde{A}, \mathbf{Z})$ is known to be a truncated polynomial ring of one generator in H^λ [**3**].

Step 8. If $\partial A = \emptyset$ and $\partial B \neq \emptyset$, then B is a point, p_0, and $A = \mathrm{cutlocus}(p_0)$. Hence, there is a homeomorphism $f: (0, \pi/2) \times S^{n-1} \to M - (A \cup B)$ by $f(t, v) = \exp_{p_0} tv$.

(a) If $\pi_1(M) = 0$, then $\pi_1(A) = 0$ and $H^*(A, \mathbf{Z})$ is a truncated polynomial ring with one generator in $H^\lambda(A, \mathbf{Z})$, $\lambda = 2, 4$, or 8. Then one can show that (b) of Theorem 1 holds.

(b) If $\pi_1(M) \neq 0$, then $(A, g_0|A)$ is isometric to $\mathbf{R}P^{n-1}(1)$ by Steps 6, 7, and $L(p_0, q)$ being an antipodal pair. Hence, (M, g_0) is isometric to $\mathbf{R}P^n(1)$ for all n.

Step 9. If $\partial A = \partial B = \emptyset$, then, in the fiber bundle $L(p, q) \hookrightarrow \mathrm{UNB}_p = S^{n-b-1} \to A$, $L(p, q)$ is not only a topological submanifold, but it is also the orbit of a closed subgroup of $O(n - b - 1)$, defined by the normal holonomy of B, so it is smooth. Step 6 shows that the fibers are equidistant: $\forall v \in L(p, q)$, $d(v, L(q, r)) = d(L(p, q), L(p, r))$, $\forall r \in A$. $L(p, q)$ and $L(q, p)$ are homeomorphic, so $\pi_1(A) = \pi_1(B)$. The normal cutlocus of A is B, and A is a strong deformation retract of $M - B$, and vice versa; see [**9**]. Using minimal geodesics, we construct homeomorphisms from UNA onto UNB and from UNA $\times (0, \pi/2)$ onto $M - (A \cup B)$.

(a) Case for $\pi_1(A) = 0$. Then $L(p, q)$ is connected and $\lambda \geq 2$. $\lambda \neq 8$ by Toda [**19**]. For $\lambda = 2$ or 4, the equidistant, smooth fibration of S^{n-a-1} has to be congruent to a Hopf fibration [**10, 11**]. One shows that A (similarly, B) is isometric to $\mathbf{C}P^{a/\lambda}$ or $\mathbf{H}P^{a/\lambda}$ by Step 6, and (M, g_0) is isometric to $\mathbf{C}P^k$ or $\mathbf{H}P^k$.

(b) Case for $\pi_1(A) \neq 0$. Unless $a = b = \lambda = 1$ and $n = 3$, at least one of A or B, say A, has codimension ≥ 3, and $B \hookrightarrow M$ induces $\pi_1(B) \cong \pi_1(M)$. If $n = 3$, Hamilton [**15**] gives the desired results; so we may assume $n \geq 4$. One constructs $\tilde{A}$ and $\tilde{B}$ in $\tilde{M}$ by lifting A and B by the natural covering map from the universal cover $\tilde{M}$ of M. $d(\tilde{M}, \tilde{g}_0) \geq \pi/2$.

If $\lambda \geq 2$, then a, b, and n are even. $\pi_1(M) = \mathbf{Z}_2$, and $d(\tilde{M}, \tilde{g}_0) = \pi/2$. $(\tilde{M}, \tilde{g}_0) \neq \mathbf{H}P^k$ since the isometry group of $\mathbf{H}P^k$ is connected for $k > 1$ [**20**] and M is unoriented [**5**, p. 98]. By Step (a) $(\tilde{M}, \tilde{g}_0)$ is isometric to $\mathbf{C}P^k$. k is odd since M is unoriented and $H^*(\mathbf{C}P^k, \mathbf{Z}) = \mathbf{Z}[x]/x^{k+1}$.

If $\lambda = 1$ that is, $L(p, q)$ is discrete, then n is even implies that $L(p, q) \simeq S^0$ and A, B are isometric to $\mathbf{R}P^a$, $\mathbf{R}P^b$, respectively, which will reduce to Step 8(b). When n is odd, $\tilde{A}$ is isometric to $S^a(1)$. Hence, $d(\tilde{M}, \tilde{g}_0) = \pi$, and one shows directly that $(\tilde{M}, \tilde{g}_0)$ is isometric to $S^n(1)$. See [**20**] for the classification of (M, g_0).

Step 10. Since (M, g_0) satisfies the conclusions of Steps 5, 8, and 9, by the finiteness theorems, we conclude Theorem 1; if such a sequence g_n does not exist, then the conclusion follows trivially.

PROOF OF THEOREM 2. . If we follow the same steps as above, then Step 5 never occurs, and Step 9(a) gives the desired result. Step 8 is the only one to modify. We construct dual sets A_1 and A_2 in A, which is possible by hypotheses (i) and (ii). If one of A_i is not a point, then the case reduces to Step 9(a) with $k \geq 3$. If A_1 and A_2 are points, then A is a sphere and $k = 2$. Clearly, $\forall q \in A$, $L(q, p) = \mathrm{UNA}_q$, so it is a great sphere in UM_q. If $i(M, g_0) = d(M, g_0) = \pi/2$, then, $\forall r \in M$, $\{r\}$ and cutlocus(r) form dual sets as in Step 8. Hence, by [**7, 8**], and fibers being equidistant and totally geodesic, it follows that the fibration of UM_p by $L(p, q)$, $q \in$ cutlocus(p), has to be congruent to a Hopf fibration. Using Step 6, one obtains that cutlocus(p) is isometric to $S^\lambda(4)$, (M, g_0) is a symmetric space, hence C^∞, as in [**2**], and (M, g_0) is isometric to $\mathbf{C}P^2$, $\mathbf{H}P^2$, or $\mathrm{Ca}\,P^2$. The rest follows from Step 10.

The complete proofs will appear elsewhere [**6**].

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