fully realized, and many authors still pay lip service to convergence. The umbral calculus, being purely algebraic, was a major force in shattering the idol of convergence. Ironically, the new insight that it afforded us provides us with the hindsight to realize that, by now, it is partly superfluous, and that many of its results can be proved directly in the framework of formal power series without the intervention of "umbra". For example, Chapter 3 culminates with a theorem that is equivalent to the famed Lagrange inversion formula. The Lagrange inversion formula, traditionally belonging to analysis, is now fully realized to be a purely algebraic fact, and a very short algebraic proof can be found, for example, in Hofbauer [2].

But even if it is true, as some people claim, that everything that the umbral calculus can do can be done faster with just formal power series, nobody can deny the elegance, insight, and sheer beauty that the umbral calculus possesses, and Steve Roman's book is an excellent account of this beautiful theory.

REFERENCES

1. M. Barnabei, A. Brimi and G. Nicoletti, *Recursive matrices and umbral calculus*, J. Algebra **75** (1982), 546–573.

2. J. Hofbauer, *Lagrange-inversion*, Seminaire Lothargien de Combinatoire (V. Strehl, ed.), 6ème Session: 29 Septembre-1 Octobre 1982, pp. 1–38.

3. S. A. Joni, *Lagrange inversion in higher dimensions and umbral operators*, Linear and Multilinear Algebra **6** (1978), 111–121.

4. J. Riordan, *Combinatorial identities*, Wiley, New York, 1968.

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Theory of function spaces, by Hans Triebel, Monographs in Mathematics, Vol. 78, Birkhäuser Verlag, Basel, 1983, 284 pp., $34.95. ISBN 3-7643-1381-1

The paradox of Besov spaces is that the very thing that makes them so successful also makes them very difficult to present and to learn. The idea behind Besov spaces begins with a simple extension of the idea of Lipschitz continuity, augmented by the observation that higher-order differences must also be used. For $s > 0$ choose any integer k greater than s. Differences are defined inductively. The first difference of a function is $f(x+h)-f(x)$ $(x \in R^n)$, and the kth difference is the composition of the $(k-1)$st and the first difference and is denoted Δ_h^k. Let $F(x, h) = \Delta_h^k f(x)/|h|^s$. The Besov space norm of f is the L^p norm of F in x followed by the L^q norm in h with respect to the measure $dh/|h|^n$. A function is in B_{pq}^s if f is in L^p and its Besov space norm is finite.

It was quickly found that there were many alternative approaches which give these same spaces. Today it is known that spaces defined by degree of approximation by entire analytic functions, spaces of functions which are values at 0 of solutions of the heat equation or Laplace's equation subject to constraints, spaces defined by spectral decomposition methods (decompose a function according to the support of its Fourier transform), spaces defined by applying fractional powers of the Laplacian (or of other semigroup generators) to the basic Lipschitz type spaces, spaces defined by interpolation from certain well-known Sobolev type spaces, and, more recently, spaces defined by a smooth "atomic" decomposition [3] can all be given norms under which they are equivalent to Besov spaces. If one has a problem to which Besov spaces may be applicable, it is very easy to choose the representation most convenient for the problem from the above list. An author has a difficult time in deciding which of the approaches to present, since to present them all would require a thousand page tome. This problem is further complicated by the fact that some of the methods have limited ranges of applicability. For example, the most intuitive, that of difference conditions, makes sense only for $s > 0$, but any problem involving duality will require negative exponents as well. Authors' tendencies have been to present the form with which they usually work and allude to the other possibilities in varying levels of detail.

Only two of the methods, the approximation method and the spectral method, are known at present to work for the full range of indices desired. To write a book that is consistent, that does not start the reader down one path and then explain why it is going to be necessary to detour down a side trail which will never intersect the main trail again, it is necessary to adopt one of these approaches, although both are counterintuitive. Moreover, in the hands of experts, both methods are powerful computationally—i.e., given a reasonably explicit function, you can decide by either method whether or not it belongs to a given space. The sharpest theorems, including [3] and the results of Bony on nonlinear differential equations, have been derived by the spectral method.

The spectral method has an additional feature that recommends it. The Littlewood-Paley characterization of L^p is seen by the spectral method to be analogous to a Besov space norm but with the orders of integration reversed. That is, in the above definition first take the L^q norm with respect to the measure $dh/|h|^n$ and follow it with the L^p norm with respect to dx. This leads one to define a new class of spaces, denoted F^s_{pq} and called the Lizorkin-Triebel spaces, which mimic the definition of Besov spaces but with the order of summation and integration reversed (all of the characterizations share this property of dependence on two parameters and operations, and, accordingly, the F spaces share the multiplicity of possible norms, although often for more restricted classes of indices). This scale allows one to see why the hope of including Sobolev spaces in the Besov scale [7] necessarily failed; they belong to the F scale instead.

With the classes B and F in hand, it is easy to show, and quite remarkable, that one has nearly all of the interesting spaces met in analysis: Lipschitz spaces, L^p spaces, H^p spaces, Sobolev spaces. There are a few holdouts—Lorentz spaces (though they could be included by adding a few more indices), the space of functions of bounded mean oscillation (which can be included if one fudges at the extreme indices, as shown by Triebel), and Morrey spaces—but perhaps one should not expect too much. It must be admitted that the F spaces have found significant application only for $q = 2$. There are also tantalizing suggestions that perhaps we are missing something—for example, when some natural operations are performed on the F spaces, one lands in a B space, and when one tries to interpolate certain B spaces, no explicit characterization can be found, although it can be shown that the answer is not a B space. Perhaps we are doing something wrong, and there is a unified approach which resolves all of these difficulties.

The present book presents the theory of B and F spaces from the point of view of the spectral method. Triebel's previous book [8] presented the spectral method but was limited to $1 < p < +\infty$, which was the state of the art at the time. Its main tool was interpolation theory. In order to incorporate results for $p \leqslant 1$ and for unifying effect, the main tools of the present book are harmonic analysis and, in particular, multiplier theorems.

The plan of the book is as follows. There are two parts: Part I comprises Chapters 2–4 and contains complete proofs. Part II, consisting of Chapters 5–10, contains a summary of results with sketches of the proofs. Chapter 2 begins with a list of interesting spaces in analysis and their usual definitions. Their common properties, particularly with respect to Fourier multipliers, are studied as heuristic motivation for the definition of B and F spaces. After the definitions are given, several alternative approaches to Besov spaces are discussed, and it is shown that all of the spaces on the list appear as either B or F spaces. Standard properties, such as inclusion theorems and trace and extension theorems, are proved, along with nonstandard results on Schauder bases and Fubini theorems for B spaces. The extension of the spaces to manifolds is studied, and the dual spaces of both classes are characterized. At the end of the chapter there is a brief discussion without proof of the approach using the heat equation and Laplace's equation. The only topic omitted is discussion of the Fourier transform images of the spaces—a theory that is not well established for the F spaces.

Chapter 3 is devoted to function spaces on domains. The approach is efficient and therefore limited. Distributions in $B(\Omega)$ are viewed as restrictions to Ω of distributions in $B(R^n)$. There are two disadvantages. First, the set Ω must be bounded and smooth (which is what you would want to have to a first approximation in any case). In fact, to allow nonsmooth domains would again require different definitions depending on the smoothness class of the domain and its geometric properties. Second, there is no inner description of the spaces —that is, we do not have a recipe, such as was given above on R^n, for what tests we should apply to a distribution to decide if it is in a B or an F space. Of course, neither the B nor the F space has an inner description for $s < 0$ even if $1 \leqslant p \leqslant \infty$. Subject to these disadvantages, essentially the whole of Chapter 2 is carried over for open domains.

In Chapter 4 these results are used to study regularity of solutions of elliptic differential equations in B or F spaces. Although this is still Part I, the results are somewhat sketchy in that the point of view is that the results (and motivation) of [8] take care of the case $1 < p < \infty$, and the concentration is on $0 < p \leqslant 1$.

Part II begins with Chapter 5, a good summary of what homogeneous spaces are and an indication of how the preceding theory carries over to them. For many problems the global constraint of f in L^p is not needed. If a problem is invariant under the dilation group on R, $t \to \delta_t$, where $\delta_t(x) = tx$, the different homogeneities of the L^p and Besov norm almost always leave one in the homogeneous spaces.

In Chapter 6 ultradistributions and weighted spaces are discussed and used in Chapter 7 to indicate results on weighted B and F spaces. Such spaces are studied on domains in Chapter 8 by the techniques of Chapter 3, and some applications to degenerate elliptic equations are given. Chapter 9 discusses periodic functions, and Chapter 10 discusses anisotropic spaces, where different degrees of smoothness are allowed in different coordinate directions.

For whom is the book written? I would think that every specialist in any of the function spaces covered would like to have a copy. The problem mentioned in the introduction confronts the nonspecialist who wishes to penetrate the area through research monographs. There are five major monographs in the area ([**1, 4, 6, 8**], and *Function spaces*). I have ignored books primarily about other subjects which have sections devoted to Besov spaces. All save [**6**] have been reviewed in the *Bulletin* [**2, 5, 7**]. Both the books of Nikol'skij [**4**] and Besov, Il'in and Nikol'skij [**1**] are rather specialized, each devoted to one method—approximation by entire functions [**4**] and integral representation formulas [**1**]. Both are restricted to $1 \leqslant p \leqslant \infty$, although it is shown in *Function spaces* that the method of [**4**] can be modified slightly so that it carries over to $p \leqslant 1$ as well. In addition, the review [**7**] of [**4**] suggests that one should read the Russian original if one is to enjoy it. The book [**1**] treats anisotropic spaces in detail. It has extensive results for domains that are not smooth and is more appropriate than *Function spaces* for those interested in these two topics.

The book by Peetre [**6**] is the best general introduction and contains wide if sketchy coverage of different approaches. It is a survey with proofs written in an informal style, including speculations, many of which are proved true in *Function spaces*. It also has a higher than normal density of misprints (they are informal lectures).

As mentioned, [**8**] is devoted exclusively to $1 < p < \infty$ and uses interpolation theory as its main tool and the spectral method as its basic approach. It has aged well although a few newer results are not included. There is adequate mention of the main alternative approaches to the theory. Both B and F spaces are covered in detail. There is also an excellent introduction to interpolation theory.

Function spaces covers the full range $0 < p \leqslant \infty$ and is up-to-date, with the exception of the approach via atomic spaces. It is rather brisk about getting to the main point, which is why I think a nonspecialist might like to have Peetre's book nearby for motivation and overall insight.

REFERENCES

1. O. V. Besov, V. P. Il'in and S. M. Nikol'skij, *Integral representations of functions and imbedding theorems*, Vols. 1, 2, V. H. Winston & Sons, Washington, D. C., 1978, 1979.

2. R. deVore, *Review of "Interpolation theory,..."*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), 339–345.