CONTINUED FRACTALS AND THE SEIFERT CONJECTURE

BY JENNY HARRISON

In 1950 Herbert Seifert posed a question today known as the Seifert Conjecture:

“Every \( C^r \) vector field on the three-sphere has either a zero or a closed integral curve.”

Paul Schweitzer published his celebrated \( C^1 \) counterexample in 1971 [Sch]. We show how to obtain a \( C^{3-\varepsilon} \) counterexample \( X \) by using techniques from number theory, analysis, fractal geometry, and differential topology [H1 and H2]. \( X \) is \( C^2 \) and its second derivative satisfies a \((1-\varepsilon)\)-Hölder condition.

1. Continued fractions and quasi-circles. Any irrational number \( \alpha \), \( 0 < \alpha < 1 \), can be expressed as a continued fraction

\[
\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}
\]

where the \( a_i \) are positive integers. One writes \( \alpha = [a_1, a_2, a_3, \ldots] \). The truncation \([a_1, \ldots, a_n] = p_n/q_n\) is the best approximation to \( \alpha \) among all rational numbers \( p/q \) with \( 0 < q \leq q_n \). The growth rate of the \( a_i \) tells “how irrational” \( \alpha = [a_i] \) is. At one extreme is the Golden Mean, \( \gamma = [1, 1, \ldots] \); at the other are Liouville numbers such as \( \lambda = [1^{1!}, 2^{2!}, 3^{3!}, \ldots] \). The former is “very irrational” while the latter is “almost rational”.

To study \( \alpha \) dynamically it is standard to consider \( R_\alpha \), the rigid rotation of the circle \( S^1 \) of unit length through angle \( \alpha \). Choose \( x \in S^1 \) and consider its \( R_\alpha \)-orbit \( O_\alpha(x) \). Since \( \alpha \) is irrational, \( O_\alpha(x) \) is dense in \( S^1 \). But how is it dense? For Liouville \( \lambda \), \( O_\lambda(x) \) contains long strings \( \{R^n_\lambda(x), R^{n+1}_\lambda(x), \ldots, R^m_\lambda(x)\} \) that are poorly distributed. They “bunch up”. In contrast, the Golden Mean’s orbit distributes itself fairly evenly throughout \( S^1 \).

Unfortunately, it is hard to distinguish visually (and hence geometrically) between bunched-up dense orbits and well distributed ones. After many iterations, the orbit picture becomes blurred. This is due in fact to the picture’s being drawn on the circle. As a remedy, we “unfold” \( S^1 \) onto a canonically constructed curve \( Q_\alpha \) in the 2-sphere \( S^2 \) as follows.

Choose a “Denjoy” projection \( \rho: S^1 \to S^1 \); that is, \( \rho \) is onto and continuous, \( \rho^{-1}(n\alpha) \) is an interval \( I_n \) for all \( n \in \mathbb{Z} \), the \( I_n \) are disjoint, and \( \rho \) is 1-1.
away from $\bigcup I_n$. By $\langle n\alpha \rangle$ we mean the fractional part of $n\alpha$. The discrepancy of $\alpha$ is

$$D_n(\alpha) = \sup_I \left\{ \frac{1}{n} \left| \sum_{m=0}^{n-1} \chi_I \langle m\alpha \rangle - |I| \right| \right\}$$

where $I$ is an interval in $S^1 = \mathbb{R}/\mathbb{Z}$ and $\chi_I$ is its characteristic function. Choose weights $w_n > 0$ so that $\sum w_n D_n(\alpha)$ converges and $w_n D_n(\alpha)$ is monotone decreasing as $|n| \to \infty$. For any $x \in S^1 \setminus I_n$, define $h_\alpha(x) = (h_1(x), h_2(x))$ in $S^1 \times \mathbb{R}$ by

$$h_1(x) = \rho(x) + \sum_{|n|=0}^{\infty} w_n (\rho(x) - \chi_{[0,\rho(x)]}(n\alpha))$$

$$h_2(x) = \sum_{|n|=0}^{\infty} \{w_{2n+1} \chi_{[0,\rho(x)]}(2n+1)\alpha - w_{2n} \chi_{[0,\rho(x)]}(2n\alpha)\}.$$
Continued Fractals and the Seifert Conjecture

\[ Q_{(\sqrt{5} - 1)/2} \]

\[ Q_{\sqrt{21}} \]

\[ Q_{(\sqrt{5} - 2)^{1/2}} \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fractals.png}
\caption{Continued Fractals.}
\end{figure}

(d) \( A^1 \supset A^2 \supset \cdots \supset A^n \supset \cdots \) and the diameters of the diamonds in \( A^n \) tend to zero as \( n \to \infty \).

(e) \( \text{diam}(\Delta_{11}) \leq \text{diam}(T_{ij}^m) \) if \( 1 \leq i, j \leq m \) and \( \Delta_{11} \) is a diagonal of \( A^n \) but not of \( A^{n-1} \).

The intersection \( C = \bigcap A^n \) is a Jordan curve consisting of all the diagonals \( \Delta_{11} \) of all the \( A^n \) plus a Cantor set \( \Gamma \),

\[ C = \Delta \cup \Gamma, \quad \Delta = \bigcup \Delta_{11}^n. \]
Such a $C$ is called a *diamond circle*. All continued fractals $Q_\alpha$ with $\alpha$ of constant type are diamond circles and all diamond circles are quasi-circles.

As a degenerate case, suppose all the diagonals of a diamond circle $C$ are points. Then $C$ is the graph of a Lipschitz function $S^1 \to \mathbb{R}$ having Lipschitz constant $\beta$ (and conversely). In the nondegenerate case and when $\alpha$ is of constant type, its continued fractal $Q = Q_\alpha$ turns out to have Hausdorff dimension $> 1$, so it cannot be a graph. Nevertheless, $Q$ has the following

**Graph-like property.** There exist angles $\eta', \eta$, $0 < \eta' < \eta < \pi$, a neighborhood $U$ of $Q$ in the cylinder, and a family of disjoint open sets $D_i$, $i \in \mathbb{Z}$, such that

(a) Each $D_i$ is a homothetic replica of a fixed hexagon and contains the interior of the diagonal $\Delta_i$. We denote $D = \bigcup D_i$.

(b) If $x \in U \setminus D$ lies on the north side of $Q$ then any point $y \in Q$ nearest $x$ lies in the downward pointing sector of angle $\eta$ at $x$. If $x$ lies near the bottom edge of $D_i$ then $y$ lies in the downward pointing sector of angle $\eta'$ at $x$. Symmetric conditions prevail south of $Q$. See Figure 3.

If $C$ is a Lipschitz graph with Lipschitz constant $\beta$ this property is obvious; $D$ is empty and $\eta = 2 \arctan \beta$. When $C$ is a general diamond curve the proof is tricky.

### 3. Denjoy homeomorphisms of $Q$ and the Whitney extension theorem.

To introduce dynamics on $Q_\alpha$ we consider any Denjoy homeomorphism $D$ of $S^1$ satisfying $\rho D = R_{2\alpha} \rho$ (recall $\rho$ from §1). Then we lift $D: S^1 \to S^1$ to $f: Q_\alpha \to Q_\alpha$ via the embedding $h_\alpha: S^1 \to Q_\alpha$. Let $\varepsilon > 0$ be given. Choose a large integer $N$ and set $\alpha = [2N, 2N, \ldots]$. The right choice of weights $w_n$ in the definition of $h_\alpha$ gives

\[
\|f(x) - f(y) - (x - y)\| < C\|x - y\|^{3-\varepsilon}
\]

for some constant $C$ and all $x, y$ in the Cantor set $\Gamma$. Using only (*), the Whitney Extension Theorem [W, AR] and the fact that $Q_\alpha$ is a quasi-circle,
we find a $C^{3-\varepsilon}$ diffeomorphism $F: S^2 \to S^2$ fixing the poles of $S^2$ such that $F|\Gamma = f|\Gamma$, $F(Q_\alpha) = Q_\alpha$, $DF|\Gamma = \text{Id}$, $D^2F|\Gamma = 0$ and $F(D_i) = D_{i+2}$. In particular, $F$ is a $C^{3-\varepsilon}$ Denjoy rotation of $Q_\alpha$, cf. [Ha, Kn].

Since no quasi-circles have $C^3$ Denjoy rotations [H4], one is led to wonder if the differentiability class $C^3$ separates Seifert counterexamples from a Seifert Theorem, much as happens in KAM theory [He, M]. It is also interesting to speculate about the relation between $\varepsilon$ and the Hausdorff dimension of $\Gamma$. Our $F$ turns out to be of class $C^{3-\varepsilon}$ and our $\Gamma$ has $HD(\Gamma) = 2 - \varepsilon$, cf. [N]. Must $HD(\Gamma)$ be large if the distortion of $Df$ at $\Gamma$ is small? Cf. [H4, H5].

4. Semistability of $g$ at $Q$. We want a modification $G$ of the Whitney extension $F$ in §3 so that $G = F$ on $Q$, $G$ is a $C^{3-\varepsilon}$ diffeomorphism of $S^2$ and $Q$ is $G$-semistable: under forward $G$-iterates $Q$ attracts the north side of $S^2\setminus Q$ and the reverse holds south of $Q$.

North of $Q$ we want to push $F(x)$ closer to $Q$ than $x$ was. The crucial fact that lets us do so (in a $C^2$ fashion near $\Gamma$) is the $C^2$-flabby condition $DF|\Gamma = \text{Id}$ and $D^2F|\Gamma = 0$. The Denjoy examples of Knill [Kn], Hall [Ha] and Herman [He] do not have this property and that is what prevents their use against the Seifert Conjecture.

In Figure 3 we indicate the directions in which we push $F(x)$ toward $Q$. Since $DF|\Gamma = \text{Id}$ and $D^2F|\Gamma = 0$, such pushing meets little resistance. At this stage of the construction we use the downward-pointing sector (shaded) from the graphlike property (§2), the fact that the quasi-slope $\beta$ of $Q$ is small, and the fact that the diagonals $\Delta_i$ slope backward. Under the resulting diffeomorphism $G$, $Q$ is semistable. North of $Q$, $G(D_i) \subset D_{i+2}$, while south of $Q$, $G^{-1}(D_i) \subset D_{i-2}$. 
Seifert counterexamples and loxodromic diffeomorphisms. The diffeomorphism $G$ constructed in §4 sends some $x_0 \in U \setminus D$ north of $Q$ into $D_0$. Under $G$ its $\alpha$-limit is the north pole $N$ and its $\omega$-limit is $\Gamma$. Similarly, some $y_0 \in D_0$ south of $Q$ is sent into $U \setminus D$ by $G$; its $\alpha$-limit is $\Gamma$ and its $\omega$-limit is the south pole $S$. Compose $G$ with a $C^\infty$ motion $M$ of $S^2$ such that $M(G(x_0)) = y_0$ and $M$ leaves all points of $S^2 \setminus D_0$ fixed. The resulting $C^{3-\varepsilon}$ diffeomorphism $H = M \circ G : S^2 \to S^2$ has the following properties:

(a) The only periodic points of $H$ are its fixed-point poles, $N$ and $S$. They are a source and sink, respectively.
(b) $\lim_{n \to -\infty} H^n(x_0) = N$ and $\lim_{n \to \infty} H^n(x_0) = S$ for some $x_0$.
(c) $H$ has a minimal set other than the poles.

(a) follows from disjointness of the $G^n(D_0)$, $n \in \mathbb{Z}$; (b) is by construction; (c) is clear—$\Gamma$ is the minimal set.

A suspension similar in spirit to Schweitzer's [Sch] lets us use $H$ to construct a $C^{3-\varepsilon}$ flow $\phi$ on $S^3$ with no compact orbits. By Hart's Smoothing Theorem [Ht], $\phi$ is conjugate to a flow $\psi$ whose generating vector field $X$ is also of class $C^{3-\varepsilon}$.

This vector field $X$ is a $C^{3-\varepsilon}$ counterexample to the Seifert conjecture. The same procedure applied to any $C^r$ diffeomorphisms $H : S^2 \to S^2$ obeying conditions (a), (b), (c) above would produce a counterexample to the $C^r$ Seifert Conjecture.

It is not known if $X$ is $C^2$ structurally stable. By Pugh's Closing Lemma it is not $C^1$ structurally stable [P]. Any diffeomorphism of $S^2$ obeying (a) and (b) but having no minimal set except the poles is topologically conjugate to the standard loxodromic diffeomorphism $z \to \frac{1}{2}z$ of the closed complex plane $C \cup \infty = S^2$. Thus we put forward the

**CONJECTURE.** Every $C^3$ diffeomorphism of $S^2$ satisfying conditions (a), (b) above is loxodromic.

This is a dissipative analogue of Birkhoff's conjecture that any measure-preserving diffeomorphism of $S^2$ whose only periodic points are the two fixed point poles must be topologically conjugate to a rigid irrational rotation of $S^2$.

**REFERENCES**


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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, EVANS HALL, BERKELEY, CALIFORNIA 94720