BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 13, Number 2, October 1985

CONTINUED FRACTALS AND THE SEIFERT CONJECTURE

BY JENNY HARRISON[1]

In 1950 Herbert Seifert posed a question today known as the Seifert Conjecture:

"Every C^r vector field on the three-sphere has either a zero or a closed integral curve."

Paul Schweitzer published his celebrated C^1 counterexample in 1971 [**Sch**]. We show how to obtain a $C^{3-\varepsilon}$ counterexample X by using techniques from number theory, analysis, fractal geometry, and differential topology [**H1** and **H2**]. X is C^2 and its second derivative satisfies a $(1-\varepsilon)$-Hölder condition.

1. Continued fractions and quasi-circles. Any irrational number α, $0 < \alpha < 1$, can be expressed as a continued fraction

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where the a_i are positive integers. One writes $\alpha = [a_1, a_2, a_3, \ldots]$. The truncation $[a_1, \ldots, a_n] = p_n/q_n$ is the best approximation to α among all rational numbers p/q with $0 < q \le q_n$. The growth rate of the a_i tells "how irrational" $\alpha = [a_i]$ is. At one extreme is the Golden Mean, $\gamma = [1, 1, \ldots]$); at the other are Liouville numbers such as $\lambda = [1^{1!}, 2^{2!}, 3^{3!}, \ldots]$. The former is "very irrational" while the latter is "almost rational".

To study α dynamically it is standard to consider R_α, the rigid rotation of the circle S^1 of unit length through angle α. Choose $x \in S^1$ and consider its R_α-orbit $O_\alpha(x)$. Since α is irrational, $O_\alpha(x)$ is dense in S^1. But how is it dense? For Liouville λ, $O_\lambda(x)$ contains long strings $\{R_\lambda^n(x), R_\lambda^{n+1}(x), \ldots, R_\lambda^m(x)\}$ that are poorly distributed. They "bunch up". In contrast, the Golden Mean's orbit distributes itself fairly evenly throughout S^1.

Unfortunately, it is hard to distinguish visually (and hence geometrically) between bunched-up dense orbits and well distributed ones. After many iterates, the orbit picture becomes blurred. This is due in fact to the picture's being drawn on the circle. As a remedy, we "unfold" S^1 onto a canonically constructed curve Q_α in the 2-sphere S^2 as follows.

Choose a "Denjoy" projection $\rho\colon S^1 \to S^1$; that is, ρ is onto and continuous, $\rho^{-1}\langle n\alpha\rangle$ is an interval I_n for all $n \in \mathbf{Z}$, the I_n are disjoint, and ρ is 1-1

Received by the editors January 13, 1982 and, in revised form, April 25, 1985.
1980 *Mathematics Subject Classification*. Primary 58F25, 34C35, 10K10.
[1]Partially supported by NSF grant no. MCS-8302062.

away from $\bigcup I_n$. By $\langle n\alpha\rangle$ we mean the fractional part of $n\alpha$. The *discrepancy* of α is

$$D_n(\alpha) = \sup_I \left\{ \frac{1}{n} \left| \sum_{m=0}^{n-1} \chi_I \langle m\alpha\rangle - |I| \right| \right\}$$

where I is an interval in $S^1 = \mathbf{R}/\mathbf{Z}$ and χ_I is its characteristic function. Choose weights $w_n > 0$ so that $\sum w_n D_n(\alpha)$ converges and $w_n D_n(\alpha)$ is monotone decreasing as $|n| \to \infty$. For any $x \in S^1 \setminus \bigcup I_n$ define $h_\alpha(x) = (h_1(x), h_2(x))$ in $S^1 \times \mathbf{R}$ by

$$h_1(x) = \rho(x) + \sum_{|n|=0}^{\infty} w_n(\rho(x) - \chi_{[0,\rho(x))}\langle n\alpha\rangle)$$

$$h_2(x) = \sum_{|n|=0}^{\infty} \{w_{2n+1}\chi_{[0,\rho(x))}\langle (2n+1)\alpha\rangle - w_{2n}\chi_{[0,\rho(x))}\langle 2n\alpha\rangle\}.$$

The mapping h_α is uniformly continuous. Its extension to the closure of $S^1 \setminus \bigcup I_n$ sends the endpoints of I_n onto points p_n, q_n in the cylinder $S^1 \times \mathbf{R}^1$ joined by a line segment Δ_n of slope ± 1. Extend h_α to S^1 so it sends I_n onto Δ_n homeomorphically.

The curve $Q_\alpha = h_\alpha(S^1)$ is the *continued fractal* corresponding to α. It depends only on α and the choice of weights; different Denjoy projections just give it different parametrizations. When α is of constant type (its a_i are bounded) we take w_n of the form $1/(1+|n|^\mu)$ with $\frac{1}{2} < \mu < 1$. In that case, Q_α turns out to be an Ahlfors quasi-circle [**Ah**]. In any case,

Think of the continued fractal Q_α as a picture of α.

Its geometry embodies not only the early patterns apparent from the circle rotation but also much of its long-term behavior.

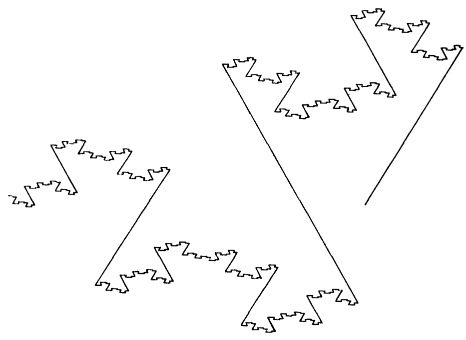
In Figure 1 are three examples drawn on the open cylinder with $x = 0$ and $x = 1$ identified. The quasi-circle Q_α with $\alpha = [4,4,4,\ldots,]$ leads to a $C^{2+\delta}$ Seifert counterexample with δ small. To raise the differentiability from $C^{2+\delta}$ to $C^{3-\varepsilon}$ we take $\alpha = [2N, 2N, \ldots]$ with N large and prove a sharpened Denjoy-Koksma inequality for such numbers α [**H6**]. The choice $\alpha = \sqrt{21}$ also leads to a $C^{2+\delta}$ example but $(\sqrt{5}-2)^{1/2}$ does not.

2. Fractal geometry of Q. When α is of constant type, the continued fractal $Q = Q_\alpha$ can be exhibited as the nested intersection of connected closed sets A^n called *β-diamond chains*, as in Figure 2. Formally, $A^n = \Delta_1^n \cup T_1^n \cup \cdots \cup \Delta_m^n \cup T_m^n$, where $m = m(n)$ and (suppressing the superscript n as appropriate)

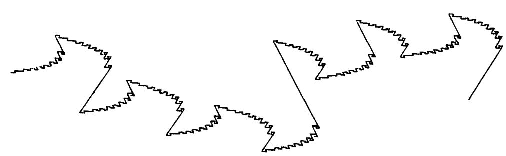
(a) Δ_i is a segment of slope ± 1 with respect to the cylinder's coordinates. The Δ_i are *diagonals* of A^n, $1 \le i \le m$.

(b) T_i is a parallelogram with interior, whose edges have slope $\pm\beta$. In practice, $\beta \ll 1$. The T_i are *β-diamonds* of A^n, $1 \le i \le m$.

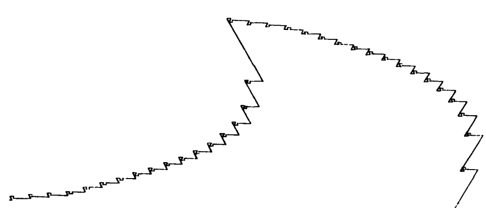
(c) Δ_i slopes backwards and joins the right-hand vertex of T_{i-1} to the left-hand vertex of T_i, $1 \le i \le m$. We call $T_0 = T_m$ to take care of the case $i = 1$.



$Q_{(\sqrt{5}-1)/2}$



$Q_{\sqrt{21}}$



$Q_{(\sqrt{5}-2)^{1/2}}$

FIGURE 1. Continued Fractals.

(d) $A^1 \supset A^2 \supset \cdots \supset A^n \supset \cdots$ and the diameters of the diamonds in A^n tend to zero as $n \to \infty$.

(e) $\operatorname{diam}(\Delta_1^n) \leq \operatorname{diam}(T_j^n)$ if $1 \leq i, j \leq m$ and Δ_1^n is a diagonal of A^n but not of A^{n-1}.

The intersection $C = \bigcap A^n$ is a Jordan curve consisting of all the diagonals Δ_1^n of all the A^n plus a Cantor set Γ,

$$C = \Delta \cup \Gamma, \qquad \Delta = \bigcup \Delta_1^n.$$

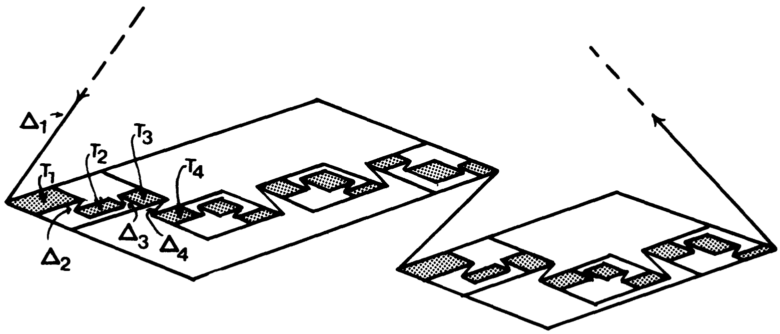


FIGURE 2. A diamond circle.

Such a C is called a *diamond circle.* All continued fractals Q_α with α of constant type are diamond circles and all diamond circles are quasi-circles.

As a degenerate case, suppose all the diagonals of a diamond circle C are points. Then C is the graph of a Lipschitz function $S^1 \to \mathbf{R}$ having Lipschitz constant β (and conversely). In the nondegenerate case and when α is of constant type, its continued fractal $Q = Q_\alpha$ turns out to have Hausdorff dimension > 1, so it cannot be a graph. Nevertheless, Q has the following

Graph-like property. There exist angles η', η, $0 < \eta' < \eta < \pi$, a neighborhood U of Q in the cylinder, and a family of disjoint open sets D_i, $i \in \mathbf{Z}$, such that

(a) Each D_i is a homothetic replica of a fixed hexagon and contains the interior of the diagonal Δ_i. We denote $D = \bigcup D_i$.

(b) If $x \in U \backslash D$ lies on the north side of Q then any point $y \in Q$ nearest x lies in the downward pointing sector of angle η at x. If x lies near the bottom edge of D_i then y lies in the downward pointing sector of angle η' at x. Symmetric conditions prevail south of Q. See Figure 3.

If C is a Lipschitz graph with Lipschitz constant β this property is obvious; D is empty and $\eta = 2\arctan\beta$. When C is a general diamond curve the proof is tricky.

3. Denjoy homeomorphisms of Q and the Whitney extension theorem. To introduce dynamics on Q_α we consider any Denjoy homeomorphism $\mathcal{D}$ of S^1 satisfying $\rho\mathcal{D} = R_{2\alpha}\rho$ (recall ρ from §1). Then we lift $\mathcal{D}\colon S^1 \to S^1$ to $f\colon Q_\alpha \to Q_\alpha$ via the embedding $h_\alpha\colon S^1 \to Q_\alpha$. Let $\varepsilon > 0$ be given. Choose a large integer N and set $\alpha = [2N, 2N, \ldots]$. The right choice of weights w_n in the definition of h_α gives

$$(*) \qquad \|f(x) - f(y) - (x - y)\| < C\|x - y\|^{3-\varepsilon}$$

for some constant C and all x, y in the Cantor set Γ. Using only $(*)$, the Whitney Extension Theorem [**W**, **AR**] and the fact that Q_α is a quasi-circle,

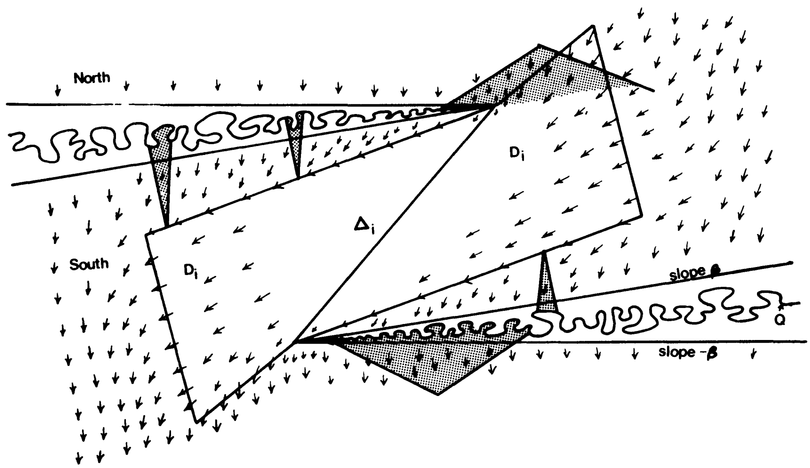


FIGURE 3. Imposing semistability.

we find a $C^{3-\varepsilon}$ diffeomorphism $F\colon S^2 \to S^2$ fixing the poles of S^2 such that $F|\Gamma = f|\Gamma$, $F(Q_\alpha) = Q_\alpha$, $DF|\Gamma = \text{Id}$, $D^2F|\Gamma = 0$ and $F(D_i) = D_{i+2}$. In particular, F is a $C^{3-\varepsilon}$ Denjoy rotation of Q_α, cf. [**Ha, Kn**].

Since no quasi-circles have C^3 Denjoy rotations [**H4**], one is led to wonder if the differentiability class C^3 separates Seifert counterexamples from a Seifert Theorem, much as happens in KAM theory [**He, M**]. It is also interesting to speculate about the relation between ε and the Hausdorff dimension of Γ. Our F turns out to be of class $C^{3-\varepsilon}$ and our Γ has $HD(\Gamma) = 2 - \varepsilon$, cf. [**N**]. Must $HD(\Gamma)$ be large if the distortion of Df at Γ is small? Cf. [**H4, H5**].

4. Semistability of g at Q. We want a modification G of the Whitney extension F in §3 so that $G = F$ on Q, G is a $C^{3-\varepsilon}$ diffeomorphism of S^2 and Q is *G-semistable*: under forward G-iterates Q attracts the north side of $S^2\backslash Q$ and the reverse holds south of Q.

North of Q we want to push $F(x)$ closer to Q than x was. The crucial fact that lets us do so (in a C^2 fashion near Γ) is the C^2-flabby condition $DF|\Gamma = \text{Id}$ and $D^2F|\Gamma = 0$. The Denjoy examples of Knill [**Kn**], Hall [**Ha**] and Herman [**He**] do not have this property and that is what prevents their use against the Seifert Conjecture.

In Figure 3 we indicate the directions in which we push $F(x)$ toward Q. Since $DF|\Gamma = \text{Id}$ and $D^2F|\Gamma = 0$, such pushing meets little resistance. At this stage of the construction we use the downward-pointing sector (shaded) from the graphlike property (§2), the fact that the quasi-slope β of Q is small, and the fact that the diagonals Δ_i slope backward. Under the resulting diffeomorphism G, Q is semistable. North of Q, $G(D_i) \subsetneqq D_{i+2}$, while south of Q, $G^{-1}(D_i) \subsetneqq D_{i-2}$.

Seifert counterexamples and loxodromic diffeomorphisms. The diffeomorphism G constructed in §4 sends some $x_0 \in U \backslash D$ north of Q into D_0. Under G its α-limit is the north pole N and its ω-limit is Γ. Similarly, some $y_0 \in D_0$ south of Q is sent into $U \backslash D$ by G; its α-limit is Γ and its ω-limit is the south pole S. Compose G with a C^∞ motion M of S^2 such that $M(G(x_0)) = y_0$ and M leaves all points of $S^2 \backslash D_0$ fixed. The resulting $C^{3-\varepsilon}$ diffeomorphism $H = M \circ G \colon S^2 \to S^2$ has the following properties:

(a) The only periodic points of H are its fixed-point poles, N and S. They are a source and sink, respectively.

(b) $\lim_{n \to -\infty} H^n(x_0) = N$ and $\lim_{n\to\infty} H^n(x_0) = S$ for some x_0.

(c) H has a minimal set other than the poles.

(a) follows from disjointness of the $G^n(D_0)$, $n \in \mathbf{Z}$; (b) is by construction; (c) is clear—Γ is the minimal set.

A suspension similar in spirit to Schweitzer's [**Sch**] lets us use H to construct a $C^{3-\varepsilon}$ flow ϕ on S^3 with no compact orbits. By Hart's Smoothing Theorem [**Ht**], ϕ is conjugate to a flow ψ whose generating vector field X is also of class $C^{3-\varepsilon}$.

> *This vector field X is a $C^{3-\varepsilon}$ counterexample to the Seifert conjecture. The same procedure applied to any C^r diffeomorphisms $H\colon S^2 \to S^2$ obeying conditions* (a), (b), (c) *above would produce a counterexample to the C^r Seifert Conjecture.*

It is not known if X is C^2 structurally stable. By Pugh's Closing Lemma it is not C^1 structurally stable [**P**]. Any diffeomorphism of S^2 obeying (a) and (b) but having no minimal set except the poles is topologically conjugate to the standard loxodromic diffeomorphism $z \to \frac{1}{2}z$ of the closed complex plane $\mathbf{C} \cup \infty = S^2$. Thus we put forward the

CONJECTURE. *Every C^3 diffeomorphism of S^2 satisfying conditions* (a), (b) *above is loxodromic.*

This is a dissipative analogue of Birkhoff's conjecture that any measure-preserving diffeomorphism of S^2 whose only periodic points are the two fixed point poles must be topologically conjugate to a rigid irrational rotation of S^2.

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