LP ESTIMATES FOR MAXIMAL FUNCTIONS AND HILBERT TRANSFORMS ALONG FLAT CONVEX CURVES IN R²

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1. Introduction and statement of results. Let \( \Gamma: \mathbb{R} \rightarrow \mathbb{R}^n \) be a curve in \( \mathbb{R}^n \) with \( \Gamma(0) = 0 \). For suitable test functions \( f \), let \( H_\Gamma f(x) = \text{p.v.} \int_{-\infty}^{\infty} f(x - \Gamma(t))t^{-1} \, dt \) and \( M_\Gamma f(x) = \sup_{0 < r < 1} |r^{-1} \int_0^r f(x - \Gamma(t)) \, dt| \). \( H_\Gamma \) and \( M_\Gamma \) are called the Hilbert transform and maximal function along \( \Gamma \), respectively. There has been considerable interest in estimates of the form \( \|H_\Gamma f\|_p \leq C\|f\|_p \) and \( \|M_\Gamma f\|_p \leq C\|f\|_p \) where \( \| \cdot \|_p \) denotes the norm in \( L^p(\mathbb{R}^n) \).

If \( \Gamma \) has some curvature at the origin, in a weak sense, then the above \( L^p \) estimates for \( H_\Gamma \) and \( M_\Gamma \) have been proved for \( 1 < p < \infty \) and \( 1 < p \leq \infty \) respectively, via techniques developed by Nagel, Riviere, Stein, and Wainger; see the survey [SW] and the references given there. More recently there has been interest in the case when \( \Gamma \) is flat to infinite order at \( t = 0 \). In particular if \( \Gamma(t) = (t, \gamma(t)) \) is a curve in \( \mathbb{R}^2 \) for which \( \gamma \) is convex for \( t > 0 \) and either even or odd, then a necessary and sufficient condition for \( H_\Gamma \) to be bounded on \( L^2 \) has been obtained in [NVWW1]. The condition for odd \( \gamma \) has also turned out to imply the \( L^2 \) boundedness of \( M_\Gamma \) [NVWW2]. There has also been progress in the study of \( L^p \) boundedness for \( p \neq 2 \) [NW, C, NVWW, C].

In the present paper we consider (locally) \( C^1 \) curves \( \Gamma(t) = (t, \gamma(t)) \) in \( \mathbb{R}^2 \) defined for \( t \geq 0 \), with \( \gamma'(0) = \gamma(0) = 0 \), convex and increasing. To discuss the Hilbert transform \( \Gamma(t) \) must be defined for \( t < 0 \); we define \( \Gamma_e(t) = (t, \gamma(-t)) \) and \( \Gamma_0(t) = (t, -\gamma(-t)) \) for \( t < 0 \). Curvature hypotheses are replaced by the much weaker “doubling property”

\[
(1.1) \quad \text{there exists } \lambda > 1 \text{ with } \gamma'(\lambda t) \geq 2\gamma'(t) \text{ for all } t > 0.
\]

We shall prove

**Theorem.** Let \( \Gamma, \Gamma_e, \Gamma_0 \) be as above and satisfy (1.1). Then \( \|M_\Gamma f\|_p \leq C\|f\|_p \) for \( 1 < p \leq \infty \), and \( \|H_{\Gamma_e} f\|_p + \|H_{\Gamma_0} f\|_p \leq C\|f\|_p \) for \( 1 < p < \infty \). More precisely, the latter assertion is that the operators \( H_\Gamma \), initially defined only for test functions, extend to bounded operators on \( L^p \).

By combining this theorem with the necessary condition for \( L^2 \) boundedness of \( H_{\Gamma_e} \) in [NVWW1], we obtain the following

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COROLLARY. For all curves $\Gamma_e$ as above, and for all $p$, $1 < p < \infty$, a necessary and sufficient condition for the boundedness of $H_{\Gamma_e}$ on $L^p$ is $(1, 1)$.

(In fact, we can see that $H_{\Gamma_e}$ is not even of weak type $(p, p)$ for any $p$, unless $(1.1)$ holds: for $0 < a < A$, let $S$ be the quadrilateral with vertices at $(\pm a, 0)$, $(-2A, \gamma(A)(-2A - a))$, $(-2A, \gamma'(A))$, $(-A, -A\gamma'(A))$, $(a - A, -A\gamma'(A))$, $(-2A + a, 0)$, $(a, -A\gamma'(A))$, $(-A, -A\gamma'(A))$, $(0, 0)$, $(a, 0)$; then $H_{\Gamma_e}(\chi_S) > \log(A/a)$ on $T$, since $\Gamma_e$ is even and convex. But, denying $(1.1)$ implies that $|S|/|T|$ can be bounded while $A/a \to \infty$.)

In previous work proofs of $L^p$ estimates of the type under discussion here have depended upon favorable decay estimates for Fourier transforms of certain measures supported on the curve $\Gamma$. In limiting cases in which $\Gamma$ consists of an infinite sequence of line segments tending to the origin such estimates fail to hold, yet $(1.1)$ may be satisfied. The principal innovation here is a Littlewood-Paley argument based on a decomposition of the Fourier transform plane into lacunary sectors as in [NSW]. A preliminary result based on this technique was proved in [CNVWW]. A similar idea was also previously used in [NSW] in studying the “lacunary” maximal function. Subsequently [DRdF] showed how old results, for cases in which favorable decay estimates do hold, could be proved by clever applications of classical Littlewood-Paley decompositions. A combination of these ideas leads to the proof of the theorem in this paper.

2. A Paley-Littlewood decomposition. Now we describe a Paley-Littlewood decomposition. Let $\alpha_k = \gamma(\lambda^k)$. Then by using the Marcinkiewicz multiplier theorem, $(1.1)$, duality, and standard techniques, we can find multiplier operators $P_k$ defined by $(P_kf)(\xi, \eta) = \Phi_k(\xi, \eta) \cdot \hat{f}(\xi, \eta)$ such that

$$ \sum_k P_k = \text{identity}; $$

$$ \text{supp} \Phi_k \subseteq \{ (\xi, \eta) : \alpha_{k-2} < |\xi/\eta| < \alpha_{k+1} \}; $$

$$ \left\| \left( \sum_k |P_kf|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p, \quad 1 < p < \infty; $$

and

$$ \left\| \sum_k P_kf_k \right\|_p \leq C_p \left\| \left( \sum_k |P_kf_k|^2 \right)^{1/2} \right\|_p, \quad 1 < p < \infty. $$

3. The proof of $\|M_{\Gamma}f\|_p \leq C\|f\|_p$ for $1 < p \leq \infty$. We may assume $\lambda \geq 2$. For each integer $k$ let $I_k$ be the interval $[\lambda^{k-1}, \lambda^k)$. Define measures $\mu_k$ by their action on test functions $\phi$: $\mu_k(\phi) = |I_k|^{-1} \int_{I_k} \phi(t, \gamma(t)) \, dt$. Then

$$ (\mu_k)(\xi, \eta) = |I_k|^{-1} \int_{I_k} \exp(i\xi t + i\eta \gamma(t)) \, dt. $$

The $L^p$ boundedness of $M_{\Gamma}$ is equivalent to

$$ \| \sup_k |\mu_k * f| \|_p \leq C\|f\|_p, \quad 1 < p \leq \infty. $$
The proof of 3.1 will be by a bootstrapping argument similar to that of [NSW]. We prove the following two lemmas:

**Lemma 1.** \[ \| \sup_k |\mu_k * f| \|_2 \leq C \| f \|_2. \] Moreover, if there exists \( r < 2 \) and \( C < \infty \) with

\[
(3.2) \quad \left\| \left( \sum_k |\mu_k * f_k|^2 \right)^{1/2} \right\|_r \leq C \left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_r
\]

for all sequences \( f_k \), then for each \( r < p < 2 \) there exists \( C_p < \infty \) such that \( \| \sup_k |\mu_k * f| \|_p \leq C_p \| f \|_p \).

**Lemma 2.** If \( \| \sup_k |\mu_k * f| \|_p \leq C_p \| f \|_p \) for some \( p, 1 < p \leq 2 \), then \( \| \left( \sum_k |\mu_k * f_k|^2 \right)^{1/2} \|_r \leq C \| \left( \sum_k |f_k|^2 \right)^{1/2} \|_r \) for all \( r \) with \( r^{-1} < (1 + p^{-1})/2 \).

3.1 follows by applying Lemmas 1 and 2 infinitely often as in [NSW]. The proof of Lemma 2 is the same as the proof of Lemma 3 of [NSW].

To prove Lemma 1 we compare \( \mu_k \) to \( \sigma_k \) where \( \sigma_k = \mu_k \star (\phi_k - \delta) \otimes (\psi_k - \delta) \). Here \( \phi(t), \psi(t) \) are nonnegative \( C^\infty \) functions on \( \mathbb{R} \) with support in \([-1,1]\) and \( \int \phi = \int \psi = 1; \phi_k(t) = \lambda^{-k} \phi(\lambda^{-k} t), \) and

\[
\psi_k(t) = [\gamma(\lambda^{k+1})]^{-1} \psi(\gamma(\lambda^{k+1}))^{-1} t.
\]

\( \delta \) is the dirac point mass at the origin. The meaning of \( (\phi_k - \delta) \otimes (\psi_k - \delta) \) is that \( \phi_k - \delta \) acts on the first variable and \( \psi_k - \delta \) on the second. We set \( \nu_k = \mu_k - \sigma_k \). Notice that

\[
\nu_k = \mu_k \star (\phi_k \otimes \delta) + \mu_k \star (\delta \otimes \psi_k) - \mu_k \star (\phi_k \otimes \psi_k)
\]

is a sum of smoothed out \( \mu_k \). One can show \( \sup_k |\nu_k * f|(x,y) \leq CM_s f(x,y) \) where \( M_s \) is the usual strong maximal function. Thus,

(3.3) \[ \| \sup_k |\nu_k * f| \|_p \leq C_p \| f \|_p, \]

(3.4) \[ \left\| \left( \sum_k |\nu_k * f_k|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_p \]

both hold for \( 1 < p \leq \infty \); see [FS].

To prove Lemma 1 it suffices to bound \( \sup_k |\sigma_k * f| \), in view of 3.3. But (letting \( P_k \) be as in §2)

\[
\sup_k |\sigma_k * f| = \sup_k \left| \sum_j \sigma_k * P_{j+k} f \right| \leq \sum_j \sup_k \left| \sigma_k * P_{j+k} f \right|
\]

\[
\leq \sum_j \left( \sum_k |\sigma_k * P_{j+k} f|^2 \right)^{1/2} \equiv \sum_j G_j f.
\]
We show
\begin{align}
||G_j f||_p & \leq C ||f||_p, \quad r < p \leq 2; \tag{3.5} \\
||G_j f||_2 & \leq C \cdot 2^{-|j|/2} ||f||_2. \tag{3.6}
\end{align}

3.5 and 3.6 imply the conclusion of Lemma 1 by a standard interpolation argument. 3.5 follows from §2, 3.2, and 3.4, 3.6 follows from the following estimates on $\tilde{\sigma}_k(\xi, \eta)$: $|\tilde{\sigma}_k(\xi, \eta)| \leq C \gamma_k|\xi|$, $|\tilde{\sigma}_k(\xi, \eta)| \leq C \gamma(\lambda^{k+1})|\eta|$, and $|\tilde{\sigma}_k(\xi, \eta)| \leq |J_k|^{-1} \max_{t \in I_k} |\xi + \eta \gamma(t)|^{-1}$.

4. The proof of $||H f||_p \leq C_p ||f||_p$, $1 < p < \infty$. The proof is similar to the proof in §3. The analogue of the operation $f \to \sigma_k * f$ is
$$L_k f = H_k \{ [\phi_k - \delta] \otimes (\psi_k - \delta) \} * f,$$
where $H_k g(x, y) = \int_{|t| \in I_k} g(x - t, y - \gamma(t)) t^{-1} \, dt$. Then we must show
$$\left\| \sum_k P_{j+k} L_k f \right\|_p \leq C ||f||_p \quad \text{and} \quad \left\| \sum_k P_{j+k} L_k f \right\|_2 \leq C \cdot 2^{-|j|/2} ||f||_2.$$

The latter follows from simple Fourier transform estimates. For the former,
$$\left\| \sum_k P_{j+k} L_k f \right\|_p \leq C \left\| \left( \sum_k \left| P_{j+k} L_k f \right|^2 \right)^{1/2} \right\|_p \leq C \left\| \left( \sum_k \left| \left[ (\phi_k - \delta) \otimes (\psi_k - \delta) \right] * P_{j+k} f \right|^2 \right)^{1/2} \right\|_p \leq C \left\| \left( \sum_k \left| P_{j+k} f \right|^2 \right)^{1/2} \right\|_p \leq C ||f||_p,$$
by §2, Lemmas 1 and 2, and [FS].

REFERENCES


Lp estimates for maximal functions


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