COMPACT RIEMANNIAN MANIFOLDS WITH POSITIVE CURVATURE OPERATORS

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The Riemann-Christoffel curvature tensor $R$ of a Riemannian manifold $M$ determines a curvature operator

$$\mathcal{R}: \Lambda^2 T_p M \to \Lambda^2 T_p M,$$

where $\Lambda^2 T_p M$ is the second exterior power of the tangent space $T_p M$ to $M$ at $p$, by the explicit formula

$$\langle \mathcal{R}(x \wedge y), z \wedge w \rangle = \langle R(x, y)w, z \rangle.$$

$M$ is said to have positive curvature operators if the eigenvalues of $\mathcal{R}$ are positive at each point $p \in M$. Meyer used the theory of harmonic forms to prove that a compact oriented $n$-dimensional Riemannian manifold with positive curvature operators must have the real homology of an $n$-dimensional sphere [GM, Proposition 2.9]. Using the theory of minimal two-spheres, we will outline a proof of the following stronger result.

**Theorem 1.** Let $M$ be a compact simply connected $n$-dimensional Riemannian manifold with positive curvature operators, where $n > 4$. Then $M$ is homeomorphic to a sphere.

Theorem 1 is actually a consequence of another theorem which makes a weaker hypothesis on the curvature tensor. To describe this hypothesis, we extend the Riemannian metric $\langle \cdot, \cdot \rangle$ in two ways to the complexified tangent space $T_p M \otimes \mathbb{C}$: as a complex symmetric bilinear form $\langle \cdot, \cdot \rangle$ and as a Hermitian inner product $\langle \langle \cdot, \cdot \rangle \rangle$. Similarly, we extend the metric in two ways to $\Lambda^2 T_p M \otimes \mathbb{C}$. An element $z \in T_p M \otimes \mathbb{C}$ is said to be isotropic if $\langle z, z \rangle = 0$. A complex linear subspace $V \subseteq T_p M \otimes \mathbb{C}$ is totally isotropic if $z \in V \Rightarrow \langle z, z \rangle = 0$.

Finally, we extend the curvature operator $\mathcal{R}$ to a complex linear map $\mathcal{R}: \Lambda^2 T_p M \otimes \mathbb{C} \to \Lambda^2 T_p M \otimes \mathbb{C}$.

**Definition.** The curvature operator $\mathcal{R}$ is positive on complex totally isotropic two-planes if whenever $\{z, w\}$ is a basis for a totally isotropic subspace of $T_p M \otimes \mathbb{C}$ of complex dimension two,

$$\langle \langle \mathcal{R}(z \wedge w), z \wedge w \rangle \rangle > 0.$$

(Note that $M$ has positive sectional curvatures if and only if its curvature operator $\mathcal{R}$ is positive on real two-planes.)

By means of a purely algebraic argument, it is possible to prove that if the sectional curvatures $K(\sigma)$ of a Riemannian manifold $M$ of dim $\geq 4$ satisfy the inequality $1/4 < K(\sigma) \leq 1$, then the curvature operator of $M$ is positive.
on complex totally isotropic two-planes. Hence the following theorem implies not only Theorem 1, but also the Rauch-Berger-Klingenberg sphere theorem for manifolds of dim \( \geq 4 \) [GKM, §7.8].

**Theorem 2.** Let \( M \) be a compact simply connected \( n \)-dimensional Riemannian manifold whose curvature operator is positive on complex totally isotropic two-planes, where \( n > 4 \). Then \( M \) is homeomorphic to a sphere.

**Sketch of Proof of Theorem 2.** Let \( S^2 = \mathbb{C} \cup \{ \infty \} \) be the Riemann sphere with the standard complex coordinate \( z = x + iy \). If \( f: S^2 \to M \) is a conformal branched minimal immersion, and \( V \) is a \( C^\infty \) section of \( f^*TM \), then the second derivative of the energy in the direction of the variation field \( V \) is given by the index form

\[
I(V, V) = \int_{S^2} \{ |\nabla_{\partial/\partial x} V|^2 + |\nabla_{\partial/\partial y} V|^2 \\
- \langle R(V, \partial f/\partial x) \partial f/\partial x, V \rangle - \langle R(V, \partial f/\partial y) \partial f/\partial y, V \rangle \} \, dx \wedge dy.
\]

It is convenient to extend this index form to a Hermitian symmetric form on \( C^\infty \) sections of \( f^*TM \otimes \mathbb{C} \). If \( W = U + iV \), where \( U \) and \( V \) are smooth sections of \( f^*TM \), then integration by parts (as in [M]) yields the formula

\[
I(W, W) = I(U, U) + I(V, V) = 4 \int_{S^2} \{ |\nabla_{\partial/\partial \bar{z}} W|^2 - \langle R(W \wedge \partial f/\partial z), W \wedge \partial f/\partial z \rangle \} \, dx \wedge dy.
\]

In this formula, \( \partial f/\partial z \) is the section of \( f^*TM \otimes \mathbb{C} \) defined by

\[
(\partial f/\partial z)(p) = (1/2)(f_*(\partial/\partial x|_p) - if_*(\partial/\partial y|_p)), \quad \text{for } p \in S^2.
\]

\( f^*TM \otimes \mathbb{C} \) can be made into a holomorphic vector bundle over \( S^2 \) in a unique fashion so that the local holomorphic sections of \( f^*TM \otimes \mathbb{C} \) are exactly the sections annihilated by \( \nabla_{\partial/\partial \bar{z}} \). When this is done, the fact that \( f \) is conformal and harmonic implies that \( \partial f/\partial z \) is an isotropic holomorphic section of \( f^*TM \otimes \mathbb{C} \). A theorem of Grothendieck [G] implies that \( f^*TM \otimes \mathbb{C} \) can be decomposed into a direct sum of holomorphic line bundles, \( f^*TM \otimes \mathbb{C} = L_1 \oplus L_2 \oplus \cdots \oplus L_n \), where

\[
c_1(L_1) \geq c_1(L_2) \geq \cdots \geq c_1(L_n), \quad c_1(L_{n-i}) = -c_1(L_i).
\]

(Here \( c_1(L_i) \) denotes the first Chern class of \( L_i \) evaluated on the fundamental cycle of \( S^2 \)). This direct sum decomposition allows us to give a lower bound on the dimension of the space of isotropic holomorphic sections of \( f^*TM \otimes \mathbb{C} \), and this bound, together with formula (2), can be used to establish

**Proposition.** If \( f: S^2 \to M \) is a nonconstant conformal branched minimal immersion into a Riemannian manifold whose curvature operator is positive on complex totally isotropic two-planes, then the index form (1) at \( f \) has index \( \geq (n/2) - (3/2) \).

(By the *index* of a symmetric bilinear form, we mean the dimension of a maximal linear subspace of the domain on which the form is negative definite.)
Now we utilize the $\alpha$-energy of Sacks and Uhlenbeck \([SU]\), the real-valued $C^2$ function on the Banach manifold $L^2_\alpha(S^2, M)$, where $\alpha$ is slightly greater than 1, defined by

$$E_\alpha(f) = \int_{S^2} (1 + |df|^2)^\alpha \, d\mu.$$  

Here $S^2$ is given the metric of constant curvature having volume one, $d\mu$ is the area element with respect to this metric, and $|df|^2$ is the energy density. We regard $M$ as isometrically imbedded in an ambient Euclidean space $E^N$, and set

$$T_f L^2_\alpha(S^2, M) = \{ V : S^2 \to E^N \mid V(p) \in T_f(p)M, \text{ for all } p \in S^2 \}.$$  

To any critical point $f$ for $E_\alpha$ is associated its Hessian, a continuous symmetric bilinear form

$$d^2 E_\alpha(f) : T_f L^2_\alpha(S^2, M) \times T_f L^2_\alpha(S^2, M) \to \mathbb{R}.$$  

**Lemma.** Let $k$ be the least integer, $2 < k < n$, such that $\pi_k(M) \neq 0$. Then there is a nonconstant critical point $f$ for $E_\alpha$ such that the Hessian of $E_\alpha$ at $f$ has index $\leq k - 2$.

Indeed, if $E_\alpha$ did not have any nonconstant critical points of index $\leq k - 2$, it could be approximated by a function whose nonconstant critical points were weakly nondegenerate (in the sense of Uhlenbeck \([U, \text{p. 432}]\)) and of index $\geq k - 1$. Then Morse theory on Banach manifolds \([U, T]\) would imply vanishing of the relative homotopy group

$$\pi_{k-2}(L^2_\alpha(S^2, M), \tilde{M}) = 0,$$

where $\tilde{M}$ is the subspace of constant maps from $S^2$ to $M$. This would contradict $\pi_k(M) \neq 0$.

By the lemma, we can choose a sequence of nonconstant critical points $f_{\alpha(i)}$ for $E_{\alpha(i)}$ of index $\leq k - 2$, with $\alpha(i) \downarrow 1$. By \([SU]\), we can assume that $E_{\alpha}(f_{\alpha}) \leq (1 + B^2)^\alpha$ and energy $(f_{\alpha}) \geq \varepsilon$, where $B$ and $\varepsilon$ are positive constants independent of $\alpha$. After passing to a subsequence, we can arrange that the $f_{\alpha(i)}$'s will $C^1$-converge on $S^2$ minus a finite number of points to a conformal branched minimal sphere \([SU, \text{Theorem 4.4}]\). If the limiting sphere is nonconstant, it can be shown that its index form has index $\leq k - 2$. (We can neglect the finite number of points at which convergence fails by an argument of Gulliver and Lawson \([GL, \text{Proposition 1.9}]\).)

If the limiting sphere is constant, then a nontrivial branched conformal minimal sphere must “bubble off” as $\alpha \to 1$ \([SU, \text{Theorem 4.6}]\). In this case, a nontrivial bubbled-off sphere must have index $\leq k - 2$.

The proposition now implies that $k - 2 \geq (n/2) - (3/2)$. Hence $\pi_i(M) = 0$, for $1 \leq i \leq n/2$. It thus follows from Poincaré duality that $M$ is a homotopy sphere, and by the resolutions of the generalized Poincaré conjecture when $n \geq 4$, $M$ must be homeomorphic to a sphere.

More details will appear in a subsequent article.

**ADDED IN PROOF.** The author has recently been informed that Micallef has independently obtained results similar to Theorems 1 and 2.
REFERENCES


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