REALIZING SYMMETRIES OF A SUBSHIFT OF FINITE TYPE
BY HOMEOMORPHISMS OF SPHERES

BY J. B. WAGONER

Let $A$ be a finite, irreducible, zero-one matrix and let $\sigma_A: X_A \rightarrow X_A$ be the corresponding subshift of finite type [F]. Recall from [F] that a Smale diffeomorphism is one with a hyperbolic zero-dimensional chain recurrent set. A well-known theorem of Williams-Smale [Wi] says that there is a Smale diffeomorphism $F_A: S^3 \rightarrow S^3$ so that $\sigma_A$ is topologically conjugate to the restriction of $F_A$ to the basic set of index one occurring as part of the spectral decomposition. Let Aut($\sigma_A$) denote the group of symmetries of $\sigma_A$—that is, the group of homeomorphisms of $X_A$ which commute with $\sigma_A$. Here is the corresponding global realization result for these symmetries.

**THEOREM.** Assume $4 < q$ and let $1 < e < q - 2$. Then there is a Smale diffeomorphism $F_A: S^q \rightarrow S^q$ with a basic set $\Omega_e$ of index $e$ (along with other basic sets of index 0, $e + 1$, $q$) together with a topological conjugacy between $\sigma_A$ and $F_A|\Omega_e$ so that, given any symmetry $g$ in Aut($\sigma_A$), there is a homeomorphism $G: S^q \rightarrow S^q$ satisfying

(A) $G$ commutes with $F_A$ on all of $S^q$,
(B) $G|\Omega_e = g$ under the identification between Aut($F_A|\Omega_e$) and Aut($\sigma_A$).

The motivation and the idea for the proof of this geometric result came by analogy from algebraic $K$-theory and pseudo-isotopy theory. The proof uses Williams’ notion of strong shift equivalence [W1, F], the contractible simplicial complex $P_A$ of topological Markov partitions for $\sigma_A$ [W1], and structural stability for Smale diffeomorphisms [R, Ro]. We would like to thank C. Pugh for useful discussions about the stability theorem.

The group Aut($\sigma_A$) is often rather large. For example, Aut($\sigma_2$) for the Bernoulli 2-shift $\sigma_2$ has been known [H] for some time to contain every finite group and to have elements of infinite order not a power of $\sigma_2$. Recently, Boyle and Lind have shown it contains the free nonabelian group on infinitely many generators. Therefore, the group of homeomorphisms of $S^q$ commuting with a certain $F_2$ is large when $4 < q$. Incidentally, at the present time not much is really known about the structure and other algebraic or homological properties of Aut($\sigma_2$). For some information see [BK] or [W1]. An open and long-standing conjecture is that Aut($\sigma_2$) is generated by $\sigma_2$ and elements of finite order.

Here is a rough idea of the proof of the Theorem. The details will appear in [W2]. Let $P$ be an $m \times m$ zero-one matrix and let $Q$ be an $n \times n$ zero-one matrix. Suppose there is an $m \times n$ zero-one matrix $R$ and an $n \times m$ zero-one

---

Received by the editors September 30, 1985.
1980 Mathematics Subject Classification (1985 Revision). Primary 34C35, 20B27.

1Partially supported by the NSF.
matrix $S$ so that $P = RS$ and $Q = SR$. As in [Wi], this determines a specific conjugacy $c_R: (X_P, \sigma_P) \rightarrow (X_Q, \sigma_Q)$ sending $x = \{x_i\}$ in $X_P$ to $c_R(x) = \{c_R(x_i)\}$ in $X_Q$, where $c_R(x_i)$ is the unique $k$ such that $R(x_i, k)S(k, x_{i+1}) = A(x_i, x_{i+1}) = 1$. Similarly for $c_S$. In fact, $c_S c_R = \sigma_P$ and $c_R c_S = \sigma_Q$, so that $c_R \sigma_P = \sigma_Q c_R$ and $c_S \sigma_Q = \sigma_P c_S$. We call $c_R$ and $c_S$ elementary symbolic conjugacies.

On the topological side, let $S^q(m)$ be the standard $q$-sphere equipped with a fixed handle decomposition with one handle of index zero, $m$ handles of index $e$, $m$ cancelling handles of index $e + 1$, and one handle of index $q$. Similarly for $S^q(n)$. One then constructs a Smale diffeomorphism $C_R: S^q(m) \rightarrow S^q(n)$ which is fitted both on the handles of index $e$ and the handles of index $e + 1$ according to the geometric intersection matrix $R$. Again, similarly for $C_S$. This is done in such a way that the composition $D_P = C_S C_R: S^q(m) \rightarrow S^q(m)$ is also a Smale diffeomorphism fitted on the $e$-handles and $(e + 1)$-handles according to the matrix $P = RS$ and $D_Q = C_R C_S: S^q(n) \rightarrow S^q(n)$ is fitted according to $Q = SR$. Observe that $C_R D_P = D_Q C_R$ and $D_P C_S = C_S D_Q$, and therefore $C_R$ and $C_S$ are smooth conjugacies between $D_P$ and $D_Q$. We call these elementary smooth conjugacies.

Now consider a Smale diffeomorphism $F_P: S^q(m) \rightarrow S^q(m)$ which is fitted on the $e$-handles and $(e + 1)$-handles by the matrix $P$. In general, of course, $F_P \neq D_P$. However, under the assumption that $1 < e < q - 2$ we are able to carefully construct $F_P$, $C_R$, and $C_S$ in such a way that there is a one-parameter family of Smale diffeomorphisms $F_P(t)$, each of which is fitted on the $e$-handles and $(e + 1)$-handles by the matrix $P$, so that $F_P(0) = F_P$, $F_P(1)$ is equal to $D_P$ on a neighborhood of the $(e + 1)$-skeleton, and both $F_P(1)$ and $D_P$ have the point at infinity as a source. Methods of stability theory [R, Ro] can then be used to produce a topological (not smooth) conjugacy between $F_P$ and $D_P$. We call this a stability conjugacy. Similarly, there is a stability conjugacy between $D_Q$ and $F_Q$, so that we then get a topological conjugacy $F_P$ and $F_Q$.

The main theorem is proved by first showing that any symmetry $g$ in $\text{Aut}(\sigma_A)$ can be obtained as the composition of a chain of elementary symbolic conjugacies and powers of shifts, and then by showing this can be mirrored compatibly with a corresponding chain of elementary smooth conjugacies, stability conjugacies, and powers of certain intermediate $F_P$ for different matrices $P$. The chain starts with the original $F_A$ which is fixed and eventually comes back to it. The composition of the various conjugacies and powers of $F_P$ in the chain give the required homeomorphism $G$.

The main theorem may well be valid on $S^4$ also, but our argument seems to require $4 < q$.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720