THE MODULI SPACE OF A PUNCTURED SURFACE AND PERTURBATIVE SERIES

R. C. PENNER

0. Introduction. Let \( F_g^s \) denote the oriented genus \( g \) surface with \( s \) punctures, \( 2g - 2 + s > 0 \), \( s \geq 1 \), and choose a distinguished puncture \( P \) of \( F_g^s \). Let \( \mathcal{T}_g^s \) be the Teichmüller space of conformal classes of complete finite-area metrics on \( F_g^s \) (see [A]), and let \( MC_g^s \) denote the mapping class group of orientation-preserving diffeomorphisms of \( F_g^s \) (fixing \( P \)) modulo isotopy (see [B]). When \( g, s \) are understood, we omit their mention. In §1 and §2, we report on joint work with D. B. A. Epstein [EP] where new and useful coordinates on \( \mathcal{T}_g^s \) are given (Theorem 2) and a \( MC_g^s \)-equivariant cell decomposition of \( \mathcal{T}_g^s \) is described (Theorem 3). There is thus an induced cell decomposition of the quotient \( M_g^s = \mathcal{T}_g^s / MC_g^s \), which is the usual moduli space of \( F_g^s \) in case \( s = 1 \). In §3, we describe a remarkable connection (see [P]) between this cell-decomposition for \( s = 1 \) and a technique from quantum field theory, which allows the computation of certain numerical invariants of \( M_g^s \) (Corollary 6). Analogues of Theorem 3 have been obtained independently by [BE and H] using different techniques. Furthermore, Corollary 7 is in agreement with some recent work in [HZ].

Let \( M \) denote Minkowski 3-space with bilinear pairing \( \langle \cdot, \cdot \rangle \) of type \((+,+,−)\), and let \( L^+ \subset M \) denote the (open) positive light-cone. The uniformization theorem (see [A]) allows us to identify \( \mathcal{T}_g^s \) with the space of (conjugacy classes of faithful and discrete) representations of \( \pi_1(F_g^s) \) in \( SO(2,1) \) (as a Fuchsian group of the first kind in the component of the identity).

I. Coordinates on \( \mathcal{T} \). Suppose \( \pi_1F = \Gamma \subset \mathcal{T} \), and choose a parabolic transformation \( \gamma \in \Gamma \) corresponding to the puncture \( P \). \( \gamma \) fixes a unique ray in \( L^+ \), and we choose a point \( z \in L^+ \) in this ray. If \( c \) is a bi-infinite geodesic in \( F \) which tends in both directions to \( P \) (to be termed simply a geodesic in the sequel), let \( \gamma(c) \in \Gamma \) denote the corresponding translation, and define the \( \lambda \)-length of (the homotopy class of) \( c \) to be \( \lambda \Gamma(c) = \sqrt{-\langle z, \gamma(c)z \rangle} \). When \( \Gamma \) is understood, we denote \( \lambda(c) = \lambda \Gamma(c) \). If \( h \) is a \( \Gamma \)-horosphere about \( P \) and \( c \) is a \( \Gamma \)-geodesic, then we define \( d_h(c) \) to be the \( \Gamma \)-hyperbolic length along \( c \) from \( h \) back to \( h \).

**Lemma 1.** If \( c_1 \) and \( c_2 \) are geodesics, then
\[
\lim_{h \to P} \exp\{d_h(c_1) - d_h(c_2)\} = [\lambda(c_1) / \lambda(c_2)]^2.
\]

It follows that \( \lambda \)-lengths are natural in the sense that if \( \varphi \in MC, \Gamma \in \mathcal{T} \), and \( c_1, c_2 \) are geodesics, then \( \lambda_{\varphi \Gamma}(c_1) / \lambda_{\varphi \Gamma}(c_2) = \lambda \Gamma(\varphi^{-1}c_1) / \lambda \Gamma(\varphi^{-1}c_2) \).

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where \( \varphi^* \) denotes push-forward of conformal type by \( \varphi \); moreover, the set of ratios of \( \lambda \)-lengths for a fixed \( \Gamma \) are discrete.

We define an ideal triangulation \( \Delta \) of \( F \) to be a decomposition of \( F \) by geodesics into regions whose doubles are thrice-punctured spheres.

**Theorem 2.** \( \lambda \)-lengths of edges of \( \Delta \) give \( \mathbb{R} \)-analytic projective coordinates on \( T \). Furthermore, \( \mathcal{M} \mathcal{C} \) acts on these coordinates by polynomials.

For the first part of the theorem, we use the \( \lambda \)-length data to build an ideal tessellation of the hyperbolic plane and apply Poincaré's Theorem to associate a conformal structure on \( F \). For the second part, we claim that the following move on ideal triangulations acts transitively: remove the diagonal \( e \) of an ideal quadrilateral \( Q \) of \( \Delta \), replacing it by the other diagonal \( f \) of \( Q \). (This well-known result follows from our Theorem 3 below.) If \((a,c), (b,d)\) are the pairs of opposite edges of \( Q \), then one computes in \( M \) that \( \lambda(e) \lambda(f) = \lambda(a) \lambda(c) + \lambda(b) \lambda(d) \). Since \( \lambda \)-lengths are natural by Lemma 1, the theorem follows.

**Remarks.** (1) The previous equation is exactly Ptolemy's theorem on side lengths of a Euclidean quadrilateral inscribed in a circle.

(2) The action of \( \mathcal{M} \mathcal{C} \) on \( \lambda \)-lengths is explicit; computer work has been done.

**II. The cell decomposition of \( \mathcal{T} \).** Consider now the orbit \( \Gamma z \) of \( z \) in \( L^+ \). Lemma 1 guarantees that \( \Gamma z \) does not accumulate in \( L^+ \) (even though the action of \( \Gamma \) on \( L^+ \) is ergodic). Furthermore, the extremal edges of the (Euclidean) convex hull of \( \Gamma z \) can be shown to project to a collection \( \Delta(\Gamma) \) of disjoint geodesics in \( F \), and regions of \( F - \Delta(\Gamma) \) are either simply connected or puncture-parallel; we call such a collection of geodesics in \( F \) an ideal cell decomposition of \( F \). Let \( \mathcal{H} \) denote the poset consisting of all

\[
C(\Delta) = \{ \Gamma \in \mathcal{T} : \Delta(\Gamma) = \Delta \}, \quad \Delta \text{ an ideal cell decomposition},
\]

where \( C(\Delta) < C(\Delta') \) if \( \Delta \subset \Delta' \).

**Theorem 3.** \( \mathcal{H} \) is an \( \mathcal{M} \mathcal{C} \)-equivariant cell decomposition of \( \mathcal{T} \). Furthermore, \( \mathcal{H} \) extends naturally to a cell decomposition of a natural compactification \( \overline{\mathcal{T}} \) of \( \mathcal{T} \) so that cells are finite-sided.

To prove Theorem 3, one first describes \( C(\Delta) \) in \( \lambda \)-length coordinates (with respect to \( \Delta \)) by a system of coupled nonlinear inequalities. Again using naturality of \( \lambda \)-lengths, we see that \( \mathcal{H} \) is an \( \mathcal{M} \mathcal{C} \)-invariant decomposition of \( \mathcal{T} \), and it remains to check that each \( C(\Delta) \) is contractible. We describe a contraction on the set of \( \lambda \)-lengths parametrizing \( C(\Delta) \), which respects the system of inequalities; this step is quite delicate and involves some estimates on side lengths of Euclidean polygons. (See Remark 1 above.) The extension to a compactification is accomplished by adjoining cells corresponding to families of geodesics disjointly embedded in \( F \) which are not ideal cell decompositions.

The duals of ideal cell decompositions of \( F^*_g \) are certain spines \( G \) in \( F^*_g \), and if \( s = 1 \), then all spines arise. We will also consider the dual spine \( \Sigma \) of \( \mathcal{H} \) on \( \mathcal{T} \). Cells of \( \Sigma \) are given by (isotopy classes of certain) pairs \((F,G)\) with
partial ordering given by Whitehead collapses of graphs. \( \Sigma \) is contractible, and \( MC \) acts on \( \Sigma \) with finite isotropy groups.

**Remark.** Suppose \( G \subset F \) is a spine of \( F \) which is dual to a cell decomposition. There is a \( \Gamma_G \in \mathcal{T} \) so that the topological symmetry group of the pair \( (F, G) \) acts as a group of conformal symmetries of \( \Gamma_G \). The corresponding matrix groups are explicitly computable and have interesting diophantine properties.

**III. Perturbative series computations.** A well-known numerical invariant of the moduli space \( M_g^1 \) of \( F_g^1 \) is the virtual (or orbifold) Euler characteristic \( \chi_g = \chi(Z)/|MC_g^1 : Z| \), where \( Z < MC_g^1 \) is finite-index torsion-free and \( \chi(Z) \) is the usual Euler characteristic. We will compute \( \chi_g \) along with a collection of further numerical invariants.

We define a **fat graph** \( G \) to be a planar projection of a graph in \( \mathbb{R}^3 \); each vertex of \( G \) is required to be at least tri-valent. A neighborhood of the vertex set inherits an orientation from \( \mathbb{R}^2 \), and we attach orientation-preserving bands along the edges of \( G \) to build a pair \( (F(G), G) \), where \( G \) is a spine of the surface \( F(G) \). We define \( v_k(G) = \# \{ k\text{-valent vertices of } G \} \), \( \Lambda(G) = \# \{ \text{boundary components of } F(G) \} \), and let \( \Gamma(G) \) (\( [G] \), resp.) denote the automorphism group (isomorphism class, resp.) of the oriented pair \( (F(G), G) \).

In what follows, we compute

\[
\phi(I, N) = \sum_{[G] \text{ with } -2\chi(G) = I} \frac{(-1)^{\Sigma v_k(G)} N^{\lambda(G)}}{\# \Gamma(G)}.
\]

The argument \( I \) determines the Euler characteristic of contributing fat graphs, and for each \( I \), \( \phi \) is a polynomial in \( N \). As a special case, the end of §2 guarantees that \( \chi_g \) is the coefficient of \( N \) in \( \phi(4g - 2, N) \) which can be taken as motivation for our interest in \( \phi \).

**Theorem 4.** We have the following equality:

\[
\phi(I, N) = \sum_{\text{tuples } v_k \text{ with } \Sigma (k-2)v_k = I} \frac{(-1)^{\Sigma v_k}}{2^{N/2} \pi^{N^2/2} \prod_{k \geq 1} v_k} \cdot \int_{M \in H^N} \prod_{k \geq 1} \left[ \text{trace } M^k \right]^{v_k} \exp \left( -\frac{1}{2} \text{ trace } M^2 \right) dM,
\]

where \( dM \) is the unitary-invariant product of Lebesgue measures

\[
dM = \left( \prod_{i=1}^{N} dM_{ii} \right) \prod_{1 \leq i < j \leq N} d(\text{Re } M_{ij}) d(\text{Im } M_{ij})
\]

on the \( N \times N \) Hermitians \( H^N \).

The technique of proof is an extension of one known amongst physicists as Feynman diagrams/perturbative series (see [BIZ]) and will not be taken up

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\(^1\)Note added in proof. The right-hand side of this equality bears a striking resemblance to Weil’s formula for the total Chern class of a bundle.
here. It is remarkable that this technique so effectively captures the combinatorics of our complex $\Sigma$, at least when $s = 1$.

**Theorem 5.** We have the following equality.

$$\phi(2I, N) = \frac{(-1)^I}{I!} \left. \frac{\partial^I}{\partial t^I} \right|_{t=0} \left[ \frac{\sqrt{2\pi t(\pi t - 1)}}{\Gamma(t-1)} \right]^N \prod_{p=1}^{N-1} (1 - pt)^{N-p}.$$  

To prove this result, one passes from the integral in Theorem 4 to an integral over $\mathbb{R}^N$. In so doing, one introduces the factor $\prod_{1 \leq i < j \leq N} (x_i - x_j)^2$ in the integrand. The integral of such a factor against $\int_{\mathbb{R}^N} d\mu(x_i)$ for some measure $d\mu$ on $\mathbb{R}$ can be expressed in terms of the zeroth moment of $d\mu$ and the coefficients of the recursion relation for the orthogonal polynomials of $d\mu$. We construct a generating function $\hat{\phi}(t, N)$ for the quantities $\phi(I, N)$ so that the resulting integral is of this form for some measure $d\mu$, whose orthogonal polynomials and moments are explicitly computable. Unfortunately, there are convergence problems for $t = 0$, and we must truncate the integrals spatially and take limits to pursue this program. Thus, $\{\phi(2I, N): I \geq 0\}$ arises as the set of coefficients in the asymptotic series at zero of the function in Theorem 5.

We define $\theta(I, N)$ similarly to $\phi(I, N)$ but summing only over connected fat graphs. Taking logarithmic derivatives and using a variant of Stirling's formula gives our main result on fat graphs.

**Corollary 6.** We have the following equality.

$$\theta(2I, N) = (-1)^I \left[ \frac{-N^{I+2}}{I(I+1)(I+2)} + \sum_{k=1}^{[(I+1)/2]} \left( \frac{I-1}{2k-2} \right) \left( \frac{B_{2k}}{2k} \right) \frac{N^{I+2-2k}}{I+2-2k} \right],$$

where $B_{2k}$ is the $2k$th Bernoulli number and $[\cdot]$ denotes integral part.

Since a fat graph with $\lambda = 1$ is necessarily connected, we find

**Corollary 7.** The virtual Euler characteristic of moduli space is

$$\chi_g = \text{coefficient of } N \text{ in } (\theta(4g - 2, N)) = \frac{-B_{2g}}{2g} = \zeta(1 - 2g),$$

where $\zeta$ is the Riemann zeta function.

Only a small part of the data about fat graphs obtained in Corollary 6 is used in Corollary 7. The remaining information is likely to be related to the action of $MC_g^\sigma$ on our decomposition of $\mathcal{T}_g^\sigma$.

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90089