PERIODIC GEODESICS OF GENERIC NONCONVEX DOMAINS IN $\mathbb{R}^2$ AND THE POISSON RELATION

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1. Introduction. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded connected domain with $C^\infty$ smooth boundary $\partial \Omega$. Consider the eigenvalues $\{\lambda_j^2\}_{j=1}^\infty$ corresponding to the Dirichlet problem for the Laplacian

$$-\Delta u = \lambda^2 u \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega.$$  

The Poisson relation for $\sigma(t) = \sum_j \cos \lambda_j t$ has the form

$$\text{singsupp}(\sigma(t)) \subset \bigcup_{\gamma \in \mathcal{L}_\Omega} \{-T_\gamma\} \cup \{0\} \cup \bigcup_{\gamma \in \mathcal{L}_\Omega} \{T_\gamma\}.$$  

Here $\mathcal{L}_\Omega$ is the union of all generalized periodic geodesics $\gamma$ in $\overline{\Omega}$, including those lying entirely on $\partial \Omega$, and $T_\gamma$ is the period (length) of $\gamma$ (see [1]).

Generalized geodesics are projections on $\overline{\Omega}$ of the generalized bicharacteristics of $\partial^2 \Omega - \Delta$, introduced by Melrose and Sjöstrand [6]. We have proved in [8, 9] that for generic strictly convex domains in $\mathbb{R}^2$ the relation (2) becomes an equality and the spectrum of (1) determines the lengths of all periodic geodesics (see [5] for related results). The purpose of this announcement is to prove the same result for generic nonconvex domains in $\mathbb{R}^2$.

2. Main results. In the analysis of (2) for nonconvex domains three difficulties appear: (A) the existence of periodic geodesics having gliding segments on $\partial \Omega$ and linear segments in the interior of $\Omega$, (B) some linear segment $l$ of a periodic geodesic could be tangent to $\partial \Omega$ at some interior point of $l$, (C) the linear Poincaré map $P_\gamma$ of a reflecting periodic geodesic $\gamma$ could contain in its spectrum 1 or $\sqrt{\mu}$ with $\mu \in \mathbb{N}$. We refer to [3] for the precise definition of reflecting geodesics and the related Poincaré map. A linear segment is a set $l = [x, y] = \{z; z = \alpha x + (1 - \alpha)y, \ 0 \leq \alpha \leq 1\}$, while a gliding segment is an arc $\delta \subset \partial \Omega$. We show below that generically for domains in $\mathbb{R}^2$ the phenomena (A), (B), (C) cannot occur. We begin by assuming $\Omega \subset \mathbb{R}^2$.

Set $\partial \Omega = X$ and consider the space $C^\infty_{\text{emb}}(X, \mathbb{R}^2)$ of all $C^\infty$ smooth embeddings of $X$ into $\mathbb{R}^2$ with the Whitney topology [2]. For $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^2)$ we denote by $\Omega_f \subset \mathbb{R}^2$ the bounded domain with boundary $f(X)$. A set $\mathcal{R} \subset C^\infty_{\text{emb}}(X, \mathbb{R}^2)$ will be called residual if $\mathcal{R}$ is a countable intersection of open dense sets.

**Theorem 1.** Let $\Omega$ be a domain with boundary $X$. There exists a residual set $\mathcal{R} \subset C^\infty_{\text{emb}}(X, \mathbb{R}^2)$ such that for every $f \in \mathcal{R}$ there are no generalized periodic geodesics $\gamma \in \mathcal{L}_\Omega$, having at least one gliding segment on $f(X)$ and
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at least one linear segment in the interior of $\Omega_f$. Moreover, for $f \in \mathcal{R}$ every reflecting geodesic $\gamma \in \mathcal{L}_\Omega_f$ has Poincaré map $P_\gamma$ whose spectrum does not contain $\sqrt{1}$ for every $p \in \mathbb{N}$.

**Remark 1.** The above result has been conjectured in [9]. For generic strictly convex domains in $\mathbb{R}^2$ the conclusion concerning Poincaré map was established by Lazutkin [4].

**Theorem 2.** Let $\Omega$ be a domain with boundary $X$. There exists a residual set $\mathcal{R} \subset C^\infty_{\text{emb}}(X, (\mathbb{R}^2)$ such that for every $f \in \mathcal{R}$ there are no generalized periodic geodesics $\gamma \in \mathcal{L}_\Omega_f$ containing at least one linear segment $l$ tangent to $f(X)$ at some interior point of $l$.

**Remark 2.** According to Theorems 1 and 2, for generic domains in $\mathbb{R}^2$ every periodic geodesic, different from the boundary, is a reflecting one. The above assertion about Poincaré map and Theorem 2 admit a generalization for domains in $\mathbb{R}^n$ which will be published elsewhere.

Combining the rational independence of periods of reflecting geodesics for generic domains, established in [8, 9], Theorems 1 and 2 and the result in [3], we obtain

**Theorem 3.** Under the assumptions and notations of Theorem 1, for every $f \in \mathcal{R}$ the Poisson relation (2) becomes an equality where $\sigma(t)$ is related to the eigenvalues for problem (1) in $\Omega_f$ with boundary condition on $f(X)$ and the unions in (2) are taken over all generalized periodic geodesics in $\mathcal{L}_\Omega_f$.

3. Idea of the proof of Theorem 1. Let $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^2)$ and let $\gamma$ be a generalized geodesic in $\mathcal{L}_\Omega_f$ having linear segments in $\mathbb{R}^2 \setminus f(X)$. Assume $\gamma$ antisymmetric, that is $\gamma$ does not contain a linear segment $l$ orthogonal to $f(X)$ at some end point of $l$. In this case there are different points $y_i = f(x_i), i = 1, \ldots, s$ on $f(X)$, an integer $k \geq s$ and a surjection $\omega: \{1, \ldots, k\} \to \{1, \ldots, s\}$ with $\omega(1) = 1, \omega(2) = 2, \omega(k) = s$, so that the linear segments $l_j = [y_\omega(j), y_\omega(j+1)], j = 1, \ldots, k - 1$ are successive segments of $\gamma$ with reflection points $y_\omega(j), j = 2, \ldots, k - 1$, the curvatures of $f(X)$ at $y_1$ and $y_k$ vanish and $l_1$ and $l_{k-1}$ are tangent to $f(X)$ at $y_1$ and $y_k$ respectively.

Setting $\omega(1) = \omega(k+1)$, we have $\omega(i) \neq \omega(i+1)$ for $i = 1, \ldots, k$ and $\{\omega(i), \omega(i+1)\} \neq \{\omega(j), \omega(j+1)\}$ whenever $1 \leq i < j \leq k - 1$. The maps having the properties listed above will be called admissible antisymmetric. Let $Z^s = \{(z_1, \ldots, z_s) \in Z^s; z_i \neq z_j$ for $i \neq j\}$. For $i = 1, \ldots, s$, set $I_i = \{j; there exists$t = 1, \ldots, k - 1$with$\{i, j\} = \{\omega(t), \omega(t + 1)\}$and denote by $U_{\omega}$ the set of those $z \in (\mathbb{R}^2)^s$ such that $z_i \notin$ convex hull $\{z_j; j \in I_i\}$ for every $i = 1, \ldots, s$. Finally, consider the map $F: U_{\omega} \to \mathbb{R}$ given by

$$F(z) = \sum_{i=1}^{k-1} \|z_{\omega(i)} - z_{\omega(i+1)}\|.$$ 

It is clear that $x' = (x_2, \ldots, x_{s-1})$ is a critical point of $F \circ f^s(x_1, x', x_s)$ considered as a function of $z' = (z_2, \ldots, z_{s-1}) \in X^{(s-1)}$, where $f^s(x) = (f(x_1), \ldots, f(x_s))$. Fix $k, s, F$ and an admissible antisymmetric map $\omega$ and denote by $T_{\omega}$ the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^2)$ such that if $x = (x_1, \ldots, x_s) \in$
$X^{(s)}$, $f^s(x) \in U_\omega$, $\text{grad}_{x'}(F \circ f^s)(x) = 0$ and the curvatures of $f(X)$ at $f(x_1)$ and $f(x_s)$ vanish, then we have $(f(x_2) - f(x_1), n_{x_1}) = 0$, $n_{x_1}$ being the normal to $f(X)$ at $x_1$ and $(\cdot, \cdot)$ the scalar product in $\mathbb{R}^3$. Our aim is to show that $T_\omega$ is residual in $C^b_{\text{emb}}(X, \mathbb{R}^2)$. To do this, we use the $s$-fold bundle of the 2-jets. Namely, let $\alpha: J^2(X, \mathbb{R}^2) \to \mathbb{R}^2$ and $\beta: J^2(X, \mathbb{R}^2) \to \mathbb{R}^2$ be the source and the target maps (see [2]). Set

$$M = \left(\alpha^{s}\right)^{-1}(X^{(s)}) \cap \left(\beta^{s}\right)^{-1}(U_\omega) \cap V,$$

where $V$ is the set of those $(j^2f_1(x_1), \ldots, j^2f_s(x_s)) \in \left(J^2(X, \mathbb{R}^2)\right)^s$ with $df_i(x_i) \neq 0$ for every $i = 1, \ldots, s$. Clearly, $M$ is an open submanifold of $J^2(X, \mathbb{R}^2) = \left(\alpha^s\right)^{-1}(X(s))$. To describe the above situation, we introduce the set $\Sigma$ of those $\sigma = (j^2f_1(x_1), \ldots, j^2f_s(x_s)) \in M$ such that $\text{grad}_{x'}(F \circ f^s)(x) = 0$, the curvature of $f_1(X)$ at $f_1(x_1)$ and that of $f_s(X)$ at $f_s(x_s)$ vanish and the vector $f_2(x_2) - f_1(x_1)$ is collinear with the tangent to $f_1(X)$ at $f_1(x_1)$. The main difficulty is to show that $\Sigma$ is a smooth submanifold of $M$ with codim $\Sigma = s + 1$. Therefore, by applying the multijet transversality theorem in [2], we prove that $T_\omega$ is residual in $C^b_{\text{emb}}(X, \mathbb{R}^2)$. Similarly we treat admissible symmetric maps $\omega$ which are related to geodesics on $f(X)$ having segments $l$ orthogonal to $f(X)$ at some end point $y \in f(X)$ of $l$. Then $\bigcap_\omega T_\omega$, where $\omega$ runs over all admissible maps, is residual in $C^b_{\text{emb}}(X, \mathbb{R}^2)$.

For the proof of the second part of Theorem 1 we use essentially the representation of Poincaré map $P_\gamma$ related to a reflecting geodesic $\gamma$, found by Petkov and Vogel [7]. We introduce a corresponding singular set $\Sigma_1$ and again the main point is to prove that $\Sigma_1$ can be covered by a countable union of smooth manifolds having codimension $s + 1$.

A similar approach is used for the proof of Theorem 2.

REFERENCES

4. V. F. Lazutkin, Convex billiard and eigenfunctions of the Laplace operator, Leningrad University, 1981. (Russian)

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