POTENTIAL THEORY FOR THE SCHRODINGER EQUATION

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Recently there has been a wave of results [2, 4, 5, 11, 15, 16, 17], on what is now referred to as the conditional gauge theorem. These works were inspired by [1 and 6]. We prove this result in greater generality than before and derive interesting new consequences. Let

\[ A = \sum \frac{\partial}{\partial x^j} \left( a_{ij}(x) \frac{\partial}{\partial x^i} \right) \]

be a uniformly elliptic operator whose coefficients are bounded measurable functions on a bounded Lipschitz domain \( D \subseteq \mathbb{R}^d \). Define the Kato class \( K_d \) as the class of functions on \( D \) such that

\[ \limsup_{\alpha \to 0} \int_{|x-y| < \alpha} \frac{|V(y)|}{|x-y|^{d-2}} \, dy = 0. \]

Our approach is to prove results about the operator \( L = A + V \) by using known results for \( A \) and studying the probabilistic quantity, the conditional gauge.

In order to introduce the conditional gauge let \( p(t, x, y) \) be the Green function for the parabolic equation \( A = \partial/\partial t \) on \( D \times (0, \infty) \). Let \( (X_t, P_x) \) be the diffusion, killed at the exit time \( \tau_D = \inf\{t > 0 : X_t \in D\} \), whose transition density is \( p(t, x, y) \). The analysis involves the diffusion \( X_t \) conditioned on its exit position. This conditioned diffusion, see [10], has transition density \( p^*(t, x, y) = K_A(x, z)^{-1} p(t, x, y) K_A(y, z) \), where \( K_A \) is the kernel function for \( A \) on \( D \), \( x, y \in D \), \( z \in \partial D \). We shall write \( P^*_x(\cdot) = P_x(\cdot | X_{\tau_D} = z) \) and \( e^V(t) = \exp\{\int_0^t V(x_s) \, ds\} \). The so-called gauge is the function on \( D \), \( F(1; x) \equiv E_x[e^V(\tau_D)] \) and the conditional gauge is defined on \( D \times \partial D \) by \( F(1; x, z) \equiv E_x^*[e^V(\tau_D)] \). Theorem 1 was first proven in [12] when \( A = \Delta, V \) is bounded and \( \partial D \) is \( C^2 \), later when \( A = \Delta, V \in K_d \) and \( \partial D \) is \( C^{1,1} \) in [16] and recently when \( A = \Delta, V \in L^p \) for some \( p > d/2 \) and \( \partial D \) is Lipschitz in [13]. Our main result is the following.

**Theorem 1.** Suppose the uniformly elliptic

\[ A = \sum \frac{\partial}{\partial x^j} \left( a_{ij}(x) \frac{\partial}{\partial x^i} \right) \]

has bounded measurable coefficients, \( V \in K_d \), and \( D \subseteq \mathbb{R}^d \) is bounded and Lipschitz. Then \( F(1; x) < \infty \) for some \( x \in D \) iff there is a positive constant \( c \) such that \( c^{-1} \leq F(1; x, z) \leq c \), \( (x, z) \in D \times \partial D \).
Sketch for Proof of Theorem 1. The proof follows [16] and requires a Green function-kernel function estimate. Let then \(G_A\) be the Green function for \(A\) and the domain \(D\). What is required are

(a) \[
\frac{G_A(x, y)K_A(y, z)}{K_A(x, z)} \leq c||x - y||^{2-d} + ||y - z||^{2-d}
\]

for some positive constant \(c\) and \(x, y \in D, z \in \partial D\), and

(b) \(E_x^z \tau_D < \infty, x \in D, z \in \partial D\).

The first involves repeated use of known estimates on \(G_A\) in terms of the Newtonian potential, an inequality due to Carleson, the boundary Harnack principle, and Harnack chain arguments, all of which are valid for \(A\) by [3]. The second follows easily from (a).

One may also condition \(X_t\) to converge to an interior point \(y \in D\) at the finite path life-time \(T\). Then by proving (a), with all \(K_A's\) replaced with \(G_A's\) and letting \(z \in D,\) we have

**Theorem 2.** \(F(1; x) < \infty\) for some \(x \in D\) if and only if there exists a positive constant \(c\) such that for all \(x, y \in D\)

\[
c^{-1} \leq F(1; x, y) = E_x^y [e_V(T)] \leq c.
\]

The next result involves the harmonic measures \(w_A\) and \(w_L\).

**Theorem 3.** Suppose \(F(1; x) < \infty\) for some \(x \in D\). Then if \(L = A + V\)

1. \(w_L^x(dz) = F(1; x, z)w_A^x(dz), (x, z) \in D \times \partial D,\)
2. \(G_L(x, y) = F(1; x, y)G_A(x, y), x, y \in D.\)

**Proof.** We discuss (1). With some work it can be shown that the Feynman-Kac formula holds. That is, the solution to the Dirichlet problem \(Lu = 0\) on \(D, u = f\) on \(\partial D\) is

\[
E_x[f(X_{\tau_D})e_V(\tau_D)] = \int_{\partial D} f(z)F(1; x, z)w_A^x(dz) = \int_{\partial D} f(z)w_L^x(dz).
\]

Thus \(F(1; x, z)w_A^x(dz) = w_L^x(dz)\). Equation (2) follows as in [17]. □

**Remark.** If \(F(1; x) < \infty\) one gets that \(w_A\) and \(w_L\) are simultaneously \(A_p\)-weights. [8 and 10] give conditions on \(A\) implying \(w_A\) is an \(A_p\)-weight relative to surface area.

We mention some consequences of Theorems 1 and 3 without proof. In general, when the gauge is finite, potential-theoretic results that hold for \(A\) and depend on bounds for \(w_A\) and \(G_A\) will also hold for \(L = A + V\). Theorem 4 was also proven in [4].

**Theorem 4 (Harnack's Inequality).** Assume \(F(1; x) < \infty\) for some \(x \in D\). There exist positive constants \(r_0\) and \(c\) such that if \(r < r_0\) and \(B(x_0, 2r) \subset D\), then for every positive solution to \(Lu = 0\) in \(D\) we have

\[
u(x) \leq cu(y), \quad x, y \in B(x_0, r).
\]

**Remark.** Harnack's inequality holds for \(A\) by [14].
Theorem 5 (Boundary Harnack Principle). Assume $F(1; x) < \infty$ for some $x \in D$. There exist positive constants $r_0$ and $c$ such that if $r < r_0$ and $z \in \partial D$ then whenever $Lu = Lv = 0$ in $D$, and $u, v$ are positive and vanish continuously on $\partial D \cap B(z, 2r)$, we have

$$\frac{u(x)}{v(x)} \leq c \frac{u(y)}{v(y)}, \quad x, y \in B(z, r) \cap D.$$ 

Remark. The boundary Harnack principle holds for $A$ by [3].

Theorem 6 (Comparison of Solutions for $A$ and $L$). Suppose $F(1; x) < \infty$ for some $x \in D$. There exist positive constants $r_0$ and $c$ such that for any $z \in \partial D$ and $r < r_0$ if $u$ and $v$ are positive solutions $Lu = 0$, $Af = 0$ on $D$ and vanish continuously on $\partial D \cap B(z, 2r)$ then

$$\frac{u(x)}{f(x)} \leq c \frac{u(y)}{f(y)}, \quad x, y \in B(z, r) \cap D.$$ 

Theorem 7 (Martin Representation). If $F(1; x) < \infty$ for some $x \in D$ then the Martin boundary for $L$ on $D$ is $\partial D$ and every positive solution to $Lu = 0$ in $D$ has the representation

$$u(x) = \int_{\partial D} K_L(x, z) \mu(dz)$$

where $K_L(x, z) = (F(1; x, z)/F(1; x_0, z))K_A(x, z)$ and $K_A(x_0, z) = 1$.

Theorem 8 (Regularity of Boundary Points). Suppose $F(1; x) < \infty$ for some $x \in D$. Then $z \in \partial D$ is regular for $L$ whenever it is regular for $A$.

Remark. This uses (2) of Theorem 3. By results of [13] $z \in \partial D$ is regular for $A$ if and only if it is regular for $\Delta$. Thus when $F(1; x) < \infty$, $\Delta$ and $L$ have the same regular points.

References


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