In this note we announce the answers to several questions which involve nonselfadjoint operator algebras. Detailed proofs will appear elsewhere.

We use the following notation. \( \mathcal{H} \) is a separable Hilbert space, \( \mathcal{B}(\mathcal{H}) \) is the algebra of bounded linear operators on \( \mathcal{H} \), and \( \mathcal{B}_1(\mathcal{H}) \) is the ideal of trace class operators on \( \mathcal{H} \). For \( T \in \mathcal{B}(\mathcal{H}) \), \( \{T\}' \) is the commutant of \( T \) and \( \{T\}'' \) is the double commutant of \( T \).

\( \mathcal{B}(\mathcal{H}) \) is the dual of \( \mathcal{B}_1(\mathcal{H}) \) (see [2]) so that \( \mathcal{B}(\mathcal{H}) \) has a weak * topology. \( \mathcal{A}(T) \) denotes the smallest weak * closed algebra containing \( T \) and \( I \), while \( \mathcal{W}(T) \) is the smallest weak operator closed algebra containing \( T \) and \( I \). \( \text{Lat} \ T \) is the lattice of (closed) invariant subspaces of \( T \), and \( \text{AlgLat} \ T = \{B \in \mathcal{B}(\mathcal{H}) : \text{Lat} \ T \subset \text{Lat} \ B\} \). It is elementary that \( \mathcal{A}(T) \subset \mathcal{W}(T) \subset \{T\}'' \subset \{T\}' \), that \( \mathcal{W}(T) \subset \text{AlgLat} \ T \), and that all of these sets except \( \mathcal{A}(T) \) are weakly closed algebras. Further, \( T \) is said to be reflexive if \( \mathcal{W}(T) = \text{AlgLat} \ T \).

We will consider the following questions.

**Question 1.** Does \( \mathcal{W}(T) = \{T\}' \cap \text{AlgLat} \ T \), \( \forall \ T \in \mathcal{B}(\mathcal{H}) \)?

**Question 2.** Does \( \mathcal{W}(T) = \mathcal{W}(T)'' \cap \text{AlgLat} \ T \), \( \forall \ T \in \mathcal{B}(\mathcal{H}) \)?

**Question 3.** Must \( \mathcal{T}^{(n)} \) be reflexive, \( \forall \ T \in \mathcal{B}(\mathcal{H}) \) and \( \forall n > 1 \)? (Here \( \mathcal{T}^{(n)} \) denotes the direct sum of \( n \) copies of \( T \)).

**Question 4.** If \( T_1 \) and \( T_2 \) are reflexive operators, must \( T_1 \oplus T_2 \) be reflexive?

**Question 5.** Does \( \mathcal{A}(T) = \mathcal{W}(T) \), \( \forall \ T \in \mathcal{B}(\mathcal{H}) \)?

**Question 6.** Does \( \mathcal{W}(T) \) have a separating vector, \( \forall \ T \subset \mathcal{B}(\mathcal{H}) \)?

Before stating the last question, we need some additional notation. Since \( \mathcal{W}(T) \) is weak * closed in \( \mathcal{B}(\mathcal{H}) \), \( \mathcal{W}(T) \) is a dual space, with predual \( \mathcal{W}(T)_* = \mathcal{B}_1(\mathcal{H})/\mathcal{W}(T)_\perp \). Here \( \mathcal{W}(T)_\perp \) denotes the preannihilator of \( \mathcal{W}(T) \). For each \( n \), let \( F_n \subset \mathcal{B}_1(\mathcal{H}) \) denote the set of operators of rank \( \leq n \).

**Question 7.** Is \( F_1/\mathcal{W}(T)_\perp \) dense in \( \mathcal{W}(T)_* \), \( \forall \ T \subset \mathcal{B}(\mathcal{H}) \)?

Some remarks regarding these questions are in order. There are some relations among the questions. For \( n = 1, 2, \) or \( 6 \), an affirmative answer to Question \( n \) implies an affirmative answer to Question \( n + 1 \).

Question 1 was raised independently by D. Sarason and P. Rosenthal (see [6, p. 195] and [7]). Rosenthal also asked Question 2 in [7]. In [4], J. Deddens listed several open questions, including Questions 3 and 4, concerning reflexive operators.

Question 5 has been raised by many people. The question appears in [2]. In [8], D. Westwood gave an example of an operator \( T \) so that \( \mathcal{A}(T) = \mathcal{W}(T) \) but so that the weak and weak * topologies are different on \( \mathcal{A}(T) \).

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Questions 6 and 7 were raised by D. Larson in a private communication. The motivation for the questions arose from the following. There has been intense research activity (see [1, 2, and 3], e.g.) on operators $T$ such that every weak * continuous linear functional on $\mathcal{W}(T)$ is represented by a rank one operator. (Thus $T$ satisfies $\mathcal{W}(T)_* = F_1/\mathcal{W}(T)_\perp$.) There are operators $T$ which do not have this property (see [5 and 1]), but for these operators $T$, $F_1/\mathcal{W}(T)_\perp$ is dense in $\mathcal{W}(T)_*$.

We have been able to show that all seven of these questions have a negative answer. The key to the construction of the counterexamples is the following theorem.

**Theorem.** Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces with $\dim \mathcal{K} = \infty$. Let $S$ be a weakly closed subspace of $\mathcal{B}(\mathcal{H})$. Then there is an operator $T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K} \oplus \mathcal{H})$ of form

$$T = \begin{pmatrix} 0 & P & 0 \\ 0 & W & Q \\ 0 & 0 & 0 \end{pmatrix}$$

so that $\mathcal{W}(T)$ splits as an independent direct sum: $\mathcal{W}(T) = \mathcal{B}(T) \oplus \tilde{S}$, where $\tilde{S} = \{ A \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K} \oplus \mathcal{H}) : A_{1,3} \in S$ and $A_{i,j} = 0$ if $(i,j) \neq (1,3) \}$ and $\mathcal{B}(T) = \{ A \in \mathcal{W}(T) : A_{1,3} = 0 \}$.

We now indicate how this theorem settles Question 1. Let $\mathcal{H} = \mathbb{C}^2$ and let $S$ be the set of trace zero operators on $\mathcal{H}$. Then $S$ is a transitive subspace of $\mathcal{B}(\mathcal{H})$. This means (see [1]) that $Sx = \mathcal{H}$ for all $x \in \mathcal{H}$, $x \neq 0$. Construct $T$ as in the theorem, so that $\mathcal{W}(T) = \mathcal{B}(T) \oplus \tilde{S}$. Now every $A \in \mathcal{B}(\mathcal{H})$ is nonzero only in its $(1,3)$ entry, so $AT = TA = 0$ and $A \in \{ T \}'$. Also, using transitivity of $S$, it is easy to see that $A \in \text{Alg Lat} T$. $S$ is a proper subspace, so $\mathcal{B}(\mathcal{H})$ is not contained in $\mathcal{W}(T)$ and we have a counterexample. We note that this example was motivated in part by the excellent survey of some finite dimension results which appears in the beginning of the paper [1] of E. Azoff.

It is easy to check that choosing $S = \mathcal{B}(\mathcal{H})$ in the theorem yields a counterexample to Questions 6 and 7. Some additional information on the structure of the subspace $\mathcal{B}(T)$ is required in order to give examples settling the remaining questions.

We now outline the proof of the theorem. We identify $\mathcal{K}$ with $\bigoplus_1^{\infty} \mathcal{H}$. In the matrix for $T$ let $P$ be the isometry of $\mathcal{H}$ into $\mathcal{K}$ with matrix $(1 \ 0 \ 0 \cdots)$. Let $W$ be a backward operator weighted shift with weight sequence $(w_nI)$ to be specified later. Thus $W$ has matrix $(W_{i,j})$ where $W_{n,n+1} = w_nI$, $n \geq 1$, and all other entries = 0. Let $\mathcal{C}$ be a countable weakly dense set in the unit ball of $S$. Let $(Q_n)$ be a sequence in $\mathcal{C}$ so that each $C \in \mathcal{C}$ appears infinitely often in $(Q_n)$. Since $Q$ is to be an operator from $\mathcal{K}$ to $\mathcal{H}$, we think of $Q$ as an operator matrix with one column. Let the $n$th entry of this column be $b_nQ_n$. Here we assume $b_n \neq 0 \ \forall n$ and that $(b_n) \in l^2$. This insures that $Q$ is bounded.
If \( n \geq 1 \), then
\[
T^{n+1} = \begin{pmatrix}
0 & PW^n & PW^{n-1}Q \\
0 & W^{n+1} & W^nQ \\
0 & 0 & 0
\end{pmatrix}.
\]

Now \( PW^{n-1}Q = \lambda_n Q_n \), where \( \lambda_n = w_1w_2 \cdots w_{n-1}b_n \). Consider the sequence \( (1/\lambda_n)T^{n+1} \). If the weights \( w_n \) are chosen to go to zero sufficiently quickly, then all matrix entries of \( (1/\lambda_n)T^{n+1} \) except for the \((1,3)\) entry go to zero with \( n \). It follows that \( \mathcal{S} \subset \mathcal{W}(T) \).

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