We consider operators in the form

\[ A = -\nabla \cdot \rho \nabla + V(x) \]

on \( \mathbb{R}^n \), where metric \( \rho = (\rho_{ij}(x)) \geq 0 \) and potential \( V(x) \geq 0 \). The classical Weyl principle for asymptotic distribution of large eigenvalues of \( A \) states that the counting function

\[ N(\lambda) = \# \{ \lambda_j \leq \lambda \} \sim \text{Vol}\{ (x; \xi) | \rho \xi \cdot \xi + V(x) \leq \lambda \} \]

as \( \lambda \to \infty \).

(See for instance [Gu].) Integrating out variable \( \xi \) we can rewrite it as

\[ N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \int (\lambda - V)^{n/2} \frac{dx}{\sqrt{\det \rho}}. \]

If potential \( V \) and metric \( \rho \) are assumed to be homogeneous in \( x \), \( V(x) = |x|^a V(x') \); \( \rho_{ij}(x) = |x|^b \rho_{ij}(x'), x' = x/|x| \), then (1) reduces to

\[ N(\lambda) \sim C \lambda^{n/2+(1-\beta/2)n/\alpha} \int V^{-\alpha/\beta}(1 - \beta/2) \frac{dS}{\sqrt{\det \rho}}; \]

integration over the unit sphere \( S \) with constant

\[ C = \frac{\omega_n}{(2\pi)^n \alpha} B \left( \frac{n}{2} + 1; \frac{n}{\alpha} (1 - \beta/2) \right), \]

which depends on the volume \( \omega_n \) of the unit sphere in \( \mathbb{R}^n \) and the beta function.

Assuming \( \beta < 2 \) we see that integral (2) becomes divergent if \( V(x') \) vanishes to a sufficiently high order. The simplest such potential is \( V(x, y) = |x|^\alpha |y|^\beta \) on \( \mathbb{R}^n + \mathbb{R}^m \).

The Weyl (volume counting) principle, when applied to the corresponding Schrödinger operator \(-\Delta + V(x)\), fails to predict discrete spectrum below any energy level \( \lambda > 0 \). However, as was shown by D. Robert [Ro] and B. Simon [Si], \( A \) has purely discrete spectrum \( \{ \lambda_j \} \to +\infty \) (for qualitative explanation of this phenomenon see [Fe]). Moreover, the “nonclassical” asymptotics of \( N(\lambda) \) was derived for such \( A \).

Recently M. Solomyak [So] studied a general class of Schrödinger operators

\[ -\Delta + V(x) \]

with homogeneous potentials \( V \) subject to the following constraint:

(A) zeros of \( V \), \( \{ x : V(x) = 0 \} \) form a smooth cone \( \Sigma \) in \( \mathbb{R}^n \) of dimension \( m \), and \( V \) vanishes on \( \Sigma \) “uniformly” to order \( b > 0 \).

Introducing variables \( x \in \Sigma \) and \( y \in N_x \) (the normal to \( \Sigma \) at \( \{ x \} \)), hypothesis (A) means that there exists

\[ \lim_{t \to 0} t^{-b} V(x + ty) = V_0(x, y). \]

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It is easy to see that $V_0(x,y)$ has mixed homogeneity

$$V_0(x,y) = |x|^a |y|^b V_0(x', y'); \quad a + b = \alpha$$

and $V_0$ approximates $V$ in a small conical neighborhood $\Sigma_\epsilon$ of $\Sigma$:

$$\Sigma_\epsilon = \{ x + y | x \in \Sigma; |y| < \epsilon |x| \}.$$

Under hypothesis (A) M. Solomyak [So] derived asymptotics of $N(\lambda)$ for such operators $A = -\Delta + V(x)$ in terms of eigenvalues $\{\lambda_j(x)\}_1^\infty$ of an auxiliary family of Schrödinger operators $\{L(x) = -\Delta_y + V_0(x, y)\}_{x \in \Sigma}$. Namely,

$$N(\lambda) \sim C \lambda \frac{m}{\pi} \left( 1 + \frac{2 + b}{a} \right) \int_{\Sigma'} \sum_{j=1}^{\infty} \lambda_j(x')^{-m(2+b)/2a} dS,$$

the integral is over $\Sigma' = \Sigma \cap \Sigma$ (unit sphere).

Notice that each operator $L(x)$ has “classical type,” so Weyl’s principle (2) applies to $\{\lambda_j(x)\}_1^\infty$,

$$\#\{\lambda_j(x) \leq \lambda\} \sim c(x) \lambda^{(n-m)(1/2+1/b)}.$$

Let us also observe that a polynomial asymptotics of $N(x) \sim c \lambda^p$ implies convergence of the series

$$\sum_{j=1}^{\infty} \lambda_j^{-q} < \infty, \text{ with any } q > p.$$

Hence by (5) the sum in (4) converges provided

$$q = m(2+b)/2a > p = (n-m)(1/2 + 1/b).$$

Condition (6) is sufficient for validity of (4). In the critical case $q = p$ an additional log $\lambda$ factor appears in (4).

The method of [So] was based on the variational formulation of the problem and certain eigenvalue estimates for Schrödinger operators in conical regions obtained in [Ros].

In the present paper we shall outline a different approach based on pseudodifferential calculus with operator-valued symbols in the spirit of [Ro]. This method allows us to recover Solomyak’s result (4) and to extend it in various directions, including operators of the form $-\nabla \cdot \rho \nabla + V(x)$.

We propose the following principle, which governs nonclassical asymptotics: the main contribution to $N(\lambda)$ comes from the degeneracy set $\Sigma$ (critical set) of $V$.

According to this principle we want to “localize” $A$ to a small (conical) neighborhood of $\Sigma$. Precisely, let us introduce the “model” operator

$$A_0 = -\Delta_\Sigma + L(x) = -\Delta_\Sigma + [-\Delta_N - 2 \nabla_x \cdot \rho' \nabla_y + V_0(x,y)]$$

on the manifold $\Sigma = \bigcup_{x \in \Sigma} N_x$, normal bundle to $\Sigma$, where $\Delta_\Sigma, \Delta_N$ are the Laplace-Beltrami operators on $\Sigma$ and the normal space, $N = N_x$, with respect to the metrics induced by $\rho_{ij}$, and $\rho'$ is the “off diagonal” part of $\rho$.

Writing $A = -\nabla \cdot \rho \nabla + V$ in normal coordinates $(x,y)$ one can show that $A = A_0$ “small perturbation” in a conical neighborhood $\Sigma_\epsilon$ of $\Sigma$. So we expect $N(\lambda; A) \sim N(\lambda; A_0)$, as $\lambda \to \infty$. 

$$A_0 = -\Delta_\Sigma + L(x) = -\Delta_\Sigma + [-\Delta_N - 2 \nabla_x \cdot \rho' \nabla_y + V_0(x,y)]$$
To study the eigenvalue distribution one usually works with certain integral "transforms" of \( N(\lambda) \), like \( \text{tr} e^{-tA} = \int_{-\infty}^{+\infty} e^{-\lambda t} dN(\lambda) \) or \( \text{tr}(\zeta + A)^{-l} = \int_{-\infty}^{+\infty} (\zeta + \lambda)^{-l} dN(\lambda) \).

We prefer to work with the latter. Once the asymptotics

\[
\text{tr}(\zeta + A)^{-l} \sim c_0 \zeta^{-l+p} \quad \text{as} \quad \zeta \to \infty
\]

is established for \( \text{tr} R^l_\zeta \) one can go back to the asymptotics of \( N(\lambda) \sim c \lambda^p \), as \( \lambda \to \infty \), by the Tauberian Theorem of M. V. Keldysh (see \[Ro\]). The relation between the two constants is \( c = c_0/pB(p; l - p) \).

So we need to establish (8).

Operator \( A \) can be thought of as a differential operator on \( \Sigma \) with operator-valued symbol \( \sum g^{ij} \xi_i \xi_j + L(x) \), where metric \( g = \rho_\Sigma - \rho^* \rho^{-1}_N \rho' \) on \( \Sigma \) is constructed from the tangent \( \rho_\Sigma \) and normal \( \rho_N \) components of \( \rho \). Then the parametrix (approximate inverse) of \( (\zeta + A_0)^{-l} \) can be constructed as an operator-valued \( \Psi \text{DO} K = K^{(l)}_\zeta \) with symbol

\[
\sigma_K = \left[ \zeta + \sum g^{ij} \xi_i \xi_j + L(x) \right]^{-l}.
\]

According to our principle we want to localize kernels \( R^l_\zeta = (\zeta + A)^{-l} \); \( \tilde{R}^l = (\zeta + A_0)^{-l} \) and \( K^{(l)}_\zeta \) to a small conical neighborhood \( \Sigma_\varepsilon \) of \( \Sigma \). Let us introduce a cut-off function

\[
\chi_\varepsilon = \begin{cases} 1 & \text{on } \Sigma_\varepsilon, \\ 0 & \text{outside}, \end{cases}
\]

and define an orthogonal projection \( P_\varepsilon u = \chi_\varepsilon u \) from \( L^2(\mathbb{R}^n) \) onto \( L^2(\Sigma_\varepsilon) \).

The following lemma plays the central role in the localization procedure.

**Lemma.** All traces below are equivalent as \( \zeta \to \infty \).

(i) \( \text{tr}(\zeta + A)^{-l} \sim \text{tr} P(\zeta + A)^{-l} P \),

(ii) \( \text{tr}(\zeta + A_0)^{-l} \sim \text{tr} P(\zeta + A_0)^{-l} P \),

(iii) \( \text{tr} K^{(l)}_\zeta \sim \text{tr} PK^{(l)}_\zeta P \),

(iv) traces of "truncated" operators : \( P(\zeta + A)^{-l} P, P(\zeta + A_0)^{-l} P, \) and \( PK^{(l)}_\zeta P \) are all equivalent.

From the lemma follows

\[
\text{tr}(\zeta + A)^{-l} \sim \text{tr} K^{(l)}_\zeta \quad \text{as} \quad \zeta \to \infty.
\]

Now it remains to compute the trace of an operator-valued \( \Psi \text{DO} K^{(l)}_\zeta \)

\[
\text{tr} K^{(l)}_\zeta = \int \int_{\Sigma} \left[ \zeta + \sum \zeta g^{ij} \xi_i \xi_j + \lambda_k(x) \right]^{-l} d\xi dx.
\]

Integrating out variables \( \xi \), using homogeneity of \( \lambda_j(x) \) and \( \rho(x) \), and introducing "polar coordinates" on \( \Sigma \) to reduce integration over the cone \( \Sigma \) to a subset \( \Sigma' = \Sigma \cap S \), we get

\[
\text{tr} K^{(l)}_\zeta = C_0 \zeta^{-l+m(1/2+\theta)} \int_\Sigma \int_{\Sigma'} \frac{\sum \lambda_j(x')^{-m\theta} dx'}{\sqrt{\det g^{ij}(x')}}
\]
with constants
\[ s = \frac{\beta b + 2a}{2 + b}; \quad \theta = \frac{1}{s}(1 - \beta/2); \quad C_0 = \int_0^\infty r^m(1-\beta/2)(1-r^s)^{m/2-1}dr. \]

Remembering that \( \{\lambda_j(x')\} \) obey the classical asymptotics (5) with exponent \( p = (n - m)(2 + b)/2b \), we obtain a sufficient condition of convergence of series (11)
\[ m\theta = \frac{m}{s}(1 - \beta/2) > \frac{(n - m)(2 + b)}{2b} \quad \text{or} \quad \frac{m(2 - \beta)}{b + 2a} > \frac{n - m}{b}. \]

Thus we have established the following

**THEOREM.** If operator \( A = -\nabla \cdot \rho \nabla + V \) with homogeneous potential \( V(x) = |x|^\alpha V(x') \geq 0 \) and nondegenerate metric \( \rho_{ij}(x) = |x|^{\beta} \rho_{ij}(x') > 0 \) satisfies hypothesis (A), then spectral function \( N(\lambda) \) of \( A \) admits the nonclassical asymptotics
\[ N(\lambda) \sim C\lambda^{m(1/2+\theta)} \int_{\Sigma'} \sum_j \lambda_j(x')^{-m\theta} \frac{dx'}{\sqrt{\det g^{ij}(x')}}. \]

provided sufficient condition (13) holds. The metric \( (g^{ij}) \) on \( \Sigma \) is obtained from components of metric \( \rho \).

**REMARKS.** (i) Formula (14) includes both the classical formula (2) with \( \beta = 0 \) and \( s = a \) (i.e., \( b = 0 \)) and all previously studied nonclassical asymptotics \([\text{Ro}, \text{Si}, \text{So}]\) (the latter corresponds to \( \beta = 0 \)).

(ii) In the critical case (equality \( m\theta = p \) in (13)) an additional \( \log \lambda \) factor appears in (16). The argument requires some modification: Before passing to the limit in the sum \( \sum_1^\infty \lambda_j^{-m\theta} \) and integration over \( \Sigma \) one has to "localize" \( K^{(i)} \) to a compact region in \( \Sigma \).

We shall illustrate our theorem and conditions by the following

**EXAMPLE.** Take scalar metric \( (\rho_{ij}) = \rho = (t^2 + |x|^2)^{\beta/2}I_{2n} \) and potential \( V = (t^2 - |x|^2)^{\beta/2} \) in the space \( \mathbb{R}^{2n} = \{(t, x): t \in \mathbb{R}; x \in \mathbb{R}^{n-1}\} \). The degeneracy set of \( V \) is the standard cone \( \Sigma = \{(t, x): t = \pm |x|\} \) in \( \mathbb{R}^n \).

Direct calculation shows: \( a = b = \alpha/2 \) and \( V_0(x, y) = |x|^{\alpha/2}|y|^{\alpha/2} \).

Condition (15) for convergence of the series of eigenvalues \( \sum_j \lambda_j^{-(n-1)\theta} \) of the operator \( L(x) = -d^2/dy^2 + |y|^{\alpha/2} \) on \( \mathbb{R} \) becomes
\[ \frac{\beta + 2}{2 - \beta} < n - 1 \quad \text{or} \quad \beta < \frac{2(n - 2)}{n}, \]
and the eigenvalue asymptotics takes a form
\[ N(\lambda) \sim C\lambda^{(n-1)(1/2+\theta)} \sum_1^\infty \lambda_j^{-(n-1)\theta} \quad \text{with} \quad \theta = \frac{(4 + \alpha)(2 - \beta)}{2\alpha(\beta + 2)}. \]

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