BOOK REVIEWS

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Non-archimedean analysis, by S. Bosch, U. Güntzer, and R. Remmert, Grundlehren der mathematischen Wissenschaften, Vol. 261, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984, xii + 436 pp., $59.00. ISBN 3-540-12546-9

Géométrie analytique rigide et applications, by Jean Fresnel and Marius van der Put, Progress in Mathematics, Vol. 18, Birkhäuser, Boston, Basel, Stuttgart, 1981, xii + 215 pp., $17.50. ISBN 3-7643-3069-4

1. Tate's rigid analytic geometry. Both books under review deal with a theory which was created in 1961 by John Tate, who at that time had given a seminar on it at Harvard and written a manuscript entitled *Rigid analytic spaces*.

These notes by Tate were distributed in Paris by the IHES in the spring of 1962 with(out) his permission and published as late as 1971 in *Inventiones Mathematicae*, whose editors thought it necessary to make these available to everyone. It is strange that the man who created this beautiful theory did nearly nothing to make it known. Further, to my knowledge he has never taken up research on the foundations of rigid analytic geometry which he has laid. I cannot guess for what reason he did not like his child later on.

I will now try to give a very rough idea of what the subject is all about. Let K be a field and $|\ |$ a *valuation* in K: a real-valued function on K for which $|0| = 0$, $|1| = 1$, $|a \cdot b| = |a| \cdot |b|$, $|a + b| \leqslant |a| + |b|$ holds for any $a, b \in K$. These valuations were introduced in order to better understand the fields of p-adic numbers constructed by Kurt Hensel in 1905. It was again Hensel who first studied p-adic analytic functions in his book *Zahlentheorie* of 1913, where he investigated properties of the p-adic exponential and logarithm. If the field K is complete with respect to the valuation, then it makes sense to single out the convergent power series. On any K-algebraic variety V one has then a natural notion of analytic functions, namely those functions which have locally convergent power series expansions in algebraic parameters. This analytic structure on V was studied in the Cartan seminar of 1960/1961. If the valuated field K is different from the field of real or complex numbers, then it is nonarchimedean, which is equivalent to being ultrametric, that is, $|a + b| \leqslant \max\{|a|, |b|\}$ is satisfied for any $a, b \in K$. In this case K is totally disconnected and the above-mentioned analytic structure fails to fulfill the basic principle of analytic continuation. As Tate says, the analytic structure gets wobbly. The main discovery of Tate consists in a procedure to define a rigid analytic structure which saves the principle of analytic continuation and gives a useful analytic theory for nonarchimedean fields K. All this is explained very nicely in the notes of Tate in 32 pages. The notes end with the definition of rigid analytic spaces.

2. The book of Bosch-Güntzer-Remmert. In the book of BGR (= Bosch-Güntzer-Remmert) a systematic approach to Tate's theory is provided in 415 pages. The book was planned in the late sixties and drafts of a large part of it existed by 1970. It consists of a long part on valuation theory and linear ultrametric analysis that should have been drastically shortened. The parts on affinoid geometry are quite brilliant provided one can appreciate the Bourbaki-type style of presenting mathematics. The word 'affinoid', whose meaning seems to be now very widely known, was suggested by R. Remmert around 1965; it is used to indicate that the affinoid spaces, which are the maximal spectra of topological algebras of finite type over K, are hybrids carrying affine algebraic as well as algebroid features. The prototype of such a space is the closed unit polydisc $\{x = (x_1, \ldots, x_n) \in K^n\colon |x_i| \leqslant 1\}$ which is the spectrum of maximal ideals of the K-algebra of strictly convergent power series in the variables $x_1, \ldots, x_n$. Here K is also assumed to be algebraically closed.

The book of BGR gives a survey of the main results of the research carried out between 1965 and 1970 by a group of persons in Göttingen and Münster, led by Grauert and Remmert. Their interest was concentrated on the abstract concepts. Among the more important results were finiteness theorems for the reduction functor and the functor of power-bounded elements, the proper mapping theorem, and the characterization of the locally closed immersions.

Rigid analytic varieties are defined in BGR using Grothendieck topologies given by systems of admissible open subsets and admissible coverings; they have to satisfy the condition that there exists an admissible covering of the entire space by affinoid subdomains. This approach is due to R. Kiehl. Unfortunately no attempt is made to relate this concept to Tate's h-structures, which are defined by selecting morphisms.

In some respects the book of BGR does not carry the subject very far. It makes no mention of differentials, derivations, or vector fields. There are not enough interesting examples of rigid analytic varieties. There are almost no indications of the more exciting developments of recent years. The application to elliptic curves on the last pages is too meager.

3. The book of Fresnel-van der Put. The book of *FP* (= Fresnel-van der Put) originated in a course given by M. van der Put in Bordeaux in 1979-1980. It proceeds very differently from that of BGR. First of all it introduces the main ideas and notions of rigid analytic geometry by studying the projective line. Then it takes on the general case of affinoid and global analytic spaces. The Grothendieck topology used seems to be much too weak; here I would advise the reader to turn to the book of BGR. The last three sections are devoted to applications; Tate curve, Néron model, stable reduction are treated, as well as analytic tori which are abelian varieties. In this respect there is more meat, and not only bones, than in BGR. For beginners the approach of FP will certainly have advantages. As a reference book the one by BGR would certainly serve better.

4. Relation to formal geometry. A question which is not touched in either book is the relation between rigid analytic geometry and Grothendieck's formal geometry. To me it seems that there is now a considerable need to clarify this relation. For instance, there is the claim of Raynaud to the effect that there exists an equivalence between the category of separable rigid analytic spaces which have a finite affinoid covering and the category of formal $\mathring{K}$-schemes of finite type localized relative to monoidal transformations with center in sheafs of ideals containing some power of the maximal ideal of $\mathring{K}$, where $\mathring{K}$ is a discrete valuation ring with quotient field K; see also [**Ma**, Chapter IV, §3, and **R**].

In 1972 David Mumford published two very important papers, [**M1**, **M2**], in which analytic constructions of degenerating curves and degenerating abelian varieties over complete rings were introduced. In both papers the essential step is the construction of a quotient space with respect to a discrete group of automorphisms. Mumford carried out the constructions in the category of formal schemes, eventually using Grothendieck's formal existence theorem to get projective varieties. An approach to this theory of nonarchimedean uniformization in the category of rigid analytic spaces is presented in [**MD**, **GP**].

These exciting new developments are only mentioned once in the introduction in the book of BGR, while some aspects are included in FP.

Recently Chai and Faltings have used the methods of [**M2**] to construct compactifications over $\mathbf{Z}$ of the moduli space of polarized abelian varieties. It should be worthwhile to express these results and arguments in the category of rigid analytic spaces also.

5. Current research. Let me finally indicate some areas of current research in which rigid analytic geometry plays an important role not mentioned in either book.

(a) DIFFERENTIAL EQUATIONS: Let X_0 be an algebraic, nonsingular variety over the field of algebraic numbers and V_0 be a locally free module sheaf with an integrable connnection ∇_0. If K is a p-adic complete field one associates with X_0 a smooth rigid analytic variety X_{rig}. As in the classical case one gets a rigid analytic connection ∇_{rig}. It has been conjectured by Baldassarri [**B**], that the canonical homomorphism

$$H^q_{DR}(X_0; (V_0, \nabla_0)) \to H^q_{DR}\big(X_{\text{rig}}; (V_{\text{rig}}, \nabla_{\text{rig}})\big)$$

is an isomorphism.

(b) MODULAR VARIETIES IN FINITE CHARACTERISTIC. Since Drinfield's important papers on elliptic modules there is a theory of modular varieties for arbitrary global function fields K_0 which is a counterpart to the classical theory of elliptic modular curves, see [**D**, **S**, **Go**]. An important variety in this context is some sort of upper half-space whose K-valued points can be described as

$$\mathbf{P}_d(K)\setminus(\text{union of all } K_\infty\text{-rational hyperplanes})$$

where K_∞ is a completion of K_0 at a prime and K a complete algebraically closed extension of K_∞.

(c) STABLE REDUCTION. M. van der Put **[P]** first used methods of rigid analytic geometry to obtain the stable reduction theorem of Deligne-Mumford. S. Bosch and W. Lütkebohmert extended these methods and proved a rigid analytic version of the semiabelian reduction theorem, first for Jacobian varieties and then for abelian varieties **[BL]**.

(d) p-ADIC ABELIAN INTEGRALS. Let C be a curve over the field $\mathbf{Q}_p$ of p-adic numbers with good reduction and ω a differential of the second kind on C. If $Z(T)$ is the numerator of the zeta function of a good reduction mod p of C, then on suitable rigid analytic open subsets X of C, $Z(\Phi^*)\omega$ is exact on X, where Φ is a lifting of Frobenius to X. Applying Dwork's principle, that the p-adic analogue of analytic continuation along a path is "analytic continuation along Frobenius," allows one to develop a global theory of p-adic integration for differentials of the second kind; see **[C]**. This theory is used to study the torsion points on curves lying on an abelian variety with complex multiplication.

(e) MODULI THEORY OF MUMFORD CURVES. The space of Mumford curves of genus g over a nonarchimedean field K can be described with the help of a nonarchimedean rigid analytic Teichmüller space of representations of the free group of rank g in $\mathrm{PGL}_2(K)$, see **[G, H]**. Also there is a nonarchimedean Siegel half-space H_g over K and an action of $\mathrm{GL}_g(\mathbf{Z})$ on H_g such that the quotient space $H_g/\mathrm{GL}_g(\mathbf{Z})$ coincides with a rigid open subdomain of the space of principally polarized abelian varieties.

(f) MUMFORD SURFACES. In 1979 Mumford gave an example of an algebraic surface using p-adic uniformization **[M3]**. There is a need for a more systematic treatment of these ideas and techniques. I have the impression that various authors are working on this question using rigid analytic geometry.

(g) p-ADIC L-FUNCTIONS. P. Schneider has presented a way of constructing p-adic L-functions using directly Mumford's theory of p-adic uniformization. If Γ is a free compact subgroup of $\mathrm{SL}_2(\mathbf{Q}_p)$ and f an automorphic form of weight n for Γ, then a p-adic L-transform μ_f is constructed and $L_p(f, s)$ is defined to be the integral $\int_{\mathbf{Z}_p^*} k^{1-s}\, d\mu_f$, where k is the canonical projection $\mathbf{Z}_p^* \to 1 \to p\mathbf{Z}_p$; see **[Sch]**.

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