SINGULAR LOCI OF SCHUBERT VARIETIES
FOR CLASSICAL GROUPS

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In this note, we give an explicit description of the singular locus of a Schubert variety in the flag variety \( G/B \), where \( G \) is a classical group, and \( B \) a Borel subgroup of \( G \).

Let \( G \) be a classical group, and \( T \) a maximal torus in \( G \). Let \( W \) be the Weyl group, and \( R \) the system of roots, of \( G \) relative to \( T \). Let \( B \) be a Borel subgroup of \( G \), where \( B \supseteq T \). Let \( S \) (resp. \( R^+ \)) be the set of simple (resp. positive) roots of \( R \) relative to \( B \). For \( \alpha \in R \), let \( s_\alpha \) be the reflection with respect to \( \alpha \), and \( X_\alpha \) the element in the Chevalley basis for the Lie algebra of \( G \), associated to \( \alpha \). For \( w \in W \), let \( e(w) \) denote the point in \( G/B \) corresponding to \( w \). The Schubert variety \( X(w) \), where \( w \in W \), is by definition the Zariski closure of \( B e(w) \) in \( G/B \). \( (X(w) \) is understood to be endowed with the canonical reduced structure.) Let \( \succeq \) denote the Bruhat order in \( W \). It is well known that for \( w_1, w_2 \in W \),

\[ w_1 \succeq w_2 \quad \text{if and only if} \quad X(w_1) \supseteq X(w_2). \]

(For generalities on algebraic groups, one may refer to [1].)

The results on the singular locus of a Schubert variety are obtained as consequences of “standard monomial theory” as developed in Geometry of \( G/P \). I–V (cf. [11, 7, 4, 5, 8]). One of the consequences of standard monomial theory is the First Basis Theorem (cf. [5, 8, 6]) which gives a \( \mathbb{Z} \) basis

\[ \{ P(\lambda, \mu), (\lambda, \mu) \text{ an admissible pair} \} \]

for \( H^0(G_Z/P_Z, L_Z) \), where \( P_Z \) is a maximal parabolic subgroup scheme of \( G_Z \) and \( L_Z \) is the ample generator of \( \text{Pic}(G_Z/P_Z) \). For any field \( k \), let us denote the canonical image of \( P(\lambda, \mu) \) in \( H^0(G_Z \otimes k/P_Z \otimes k, L_Z \otimes k) \) by \( p(\lambda, \mu) \). In [9], it is shown that over any field \( k \), for \( w, \tau \in W \), with \( w \succeq \tau \), the Zariski tangent space \( T(w, \tau) \), to \( X(w) \) at \( e(\tau) \) is spanned by

\[ \left\{ X^-_{-\beta}, \beta \in \tau(R^+) \right\} \quad \text{for all} \quad (\lambda, \mu) \quad \text{such that} \quad X^-_{-\beta} p(\lambda, \mu) = cp(\tau, \tau), c \in k^*, \]

\[ \left\{ p(\lambda, \mu) \mid_{X(w)} \neq 0 \right\} \]

Denoting by \( \{ Q(\lambda, \mu) \} \) the basis for the \( \mathbb{Z} \)-dual of \( H^0(G_Z/P_Z, L_Z) \), dual to the basis \( \{ P(\lambda, \mu) \} \), it can be seen easily that \( X^-_{-\beta} p(\lambda, \mu) = cp(\tau, \tau), c \in k^* \), if and only if \( X^-_{-\beta} Q(\tau, \tau) \), when written as a \( \mathbb{Z} \)-linear combination of the elements

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$Q(\theta, \delta)$, involves $Q(\lambda, \mu)$ with a coefficient that is nonzero in $k$. From this we obtain that $T(w, \tau)$ is spanned by

$$\left\{ X_{-\beta}, \beta \in \tau(R^+) \text{ for all } (\lambda, \mu) \text{ such that } Q(\lambda, \mu) \text{ occurs in } X_{-\beta}Q(\tau, \tau) \right\}.$$ 

From this we obtain that $T(w, \tau)$ is spanned by $\lambda \in T(R^+)$ for all $(\lambda, \mu)$ such that $Q(\lambda, \mu)$ occurs in $X_{-\beta}Q(\tau, \tau)$ with a coefficient that is nonzero in $k, w \geq \lambda$.

In [3], we have given an explicit description of $Q(\lambda, \mu)$ for the case of a classical group. Using this description, we express $X_{-\beta}Q(\tau, \tau)$ as a linear combination with integer coefficients of the $Q(\theta, \delta)$'s. This enables us to obtain an explicit description of the singular locus of $X(w)$.

Let $G$ be classical of rank $n$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$, the order being as in [2]. Further, we follow the notation in [2] to denote the elements of $R$. For $1 \leq d \leq n$, we fix the following:

$$P_d = \left\{ \text{the maximal parabolic subgroup of } G \right\}$$
$$W_{P_d} = \text{Weyl group of } P_d,$n
$$W^P = \text{the set of "minimal representatives" of } W/W_{P_d}.$$

Recall (cf. [2, 4]) that $$W^P = \{ w \in W \mid l(ws_{\alpha_i}) = l(w) + 1, \ 1 \leq i \leq n, \ i \neq d \}. $$

It is known (cf. [2]) that

$$W = W_{P_d} \cdot W^P.$$ 

For $w \in W$, let $w^{(d)}$ be the element in $W^P$ corresponding to the coset $wW_{P_d}$. We have

$$w^{(d)} = wW_{P_d} \cap W^P.$$ 

Let $$A = \{(a_1, \ldots, a_d) \mid a_1 < a_2 < \cdots < a_d, \ a_i \in \mathbb{Z}\}.$$ 

We have a natural partial order $\geq$ in $A$, namely,

$$\text{(a_1, \ldots, a_d)} \geq (b_1, \ldots, b_d), \ \text{if } a_i \geq b_i, \ 1 \leq i \leq d.$$ 

This partial order among $d$-tuples will be used in the sequel in describing the Bruhat order in $W^P$. Further, for any $d$-tuple $(z_1, \ldots, z_d)$ of integers, we let

$$\text{(z_1, \ldots, z_d)} \uparrow = (z_{i_1}, z_{i_2}, \ldots, z_{i_d})$$

where $j \rightarrow i_j$ is a permutation and $z_{i_j} \leq z_{i_{j+1}}$. Thus, $(z_1, \ldots, z_d) \uparrow$ is the $d$-tuple whose entries are obtained by arranging the entries $(z_1, \ldots, z_d)$ in increasing order. We shall denote the elements of the symmetric group $S_m$, where $m \in \mathbb{N}$, in the following way. Let $\sigma \in S_m$ be such that

$$\text{\sigma(i) = c_i, \ 1 \leq i \leq m}.$$ 

We shall denote $\sigma$ by $(c_1 \cdots c_m)$. Let $k$ be the base field. For any positive integer $m$, let $\{e_1, \ldots, e_m\}$ denote the standard basis of $k^m$. 


I. The symplectic group $\text{Sp}(2n)$. Let $E = \left( \begin{array}{cc} J & 0 \\ \cdot & \cdot \\ 0 & J \end{array} \right)_{n \times n}$, where
\[ J = \left( \begin{array}{cc} 0 & 1 \\ \cdot & \cdot \\ 1 & 0 \end{array} \right) \]

Let $(\cdot, \cdot)$ be the skew symmetric bilinear form on $k^{2n}$, represented by $E$, with respect to $\{e_1, \ldots, e_{2n}\}$. Let $G = \text{Sp}(2n) = \{A \in \text{SL}(2n) | \text{tr}(AEG = E)\}$.

Let $\sigma$ be the involution on $\text{SL}(2n)$ defined by
\[ \sigma(A) = \left( A^{-1} \right)^{\text{tr}}, \quad A \in \text{SL}(2n). \]

We see that $\text{Sp}(2n) = \text{SL}(2n)^\sigma$. In view of (8), we obtain an identification of $W$, the Weyl group of $G$, with a subgroup of $S_{2n}$ (= the Weyl group of $\text{SL}(2n)$), namely
\[ W = \{(a_1 \cdots a_{2n}) | a_i = 2n + 1 - a_{2n+1-i}, \ 1 \leq i \leq 2n\}. \]


The above identification (cf. (9)) of $W$, and straightforward calculations using the definitions of [2] allow us to identify $W^P_d$ as
\[ W^P_d = \left\{ (a_1, \ldots, a_d) \mid \begin{array}{l} (1) \ 1 \leq a_1 < a_2 \cdots < a_d \leq 2n, \\
(2) \text{for } 1 \leq i \leq 2n, \text{ if } i \in \{a_1, \ldots, a_d\}, \text{ then } 2n + 1 - i \notin \{a_1, \ldots, a_d\} \end{array} \right\}. \]

For $w \in W$, say $w = (c_1 \cdots c_{2n})$, we see easily that
\[ w^{(d)} = (c_1, \ldots, c_d) \uparrow. \]

Under the above identification of $W^P_d$, we have (cf. [10]), given two elements $(a_1, \ldots, a_d), (b_1, \ldots, b_d)$ in $W^P_d$,
\[ (a_1, \ldots, a_d) \succeq (b_1, \ldots, b_d) \text{ if and only if } (a_1, \ldots, a_d) \succeq (b_1, \ldots, b_d). \]

Thus, the Bruhat order in $W^P_d$ coincides with the natural order (cf. equation (3)) on d-tuples.

PROPOSITION C.1. Let $G = \text{Sp}(2n)$. For $1 \leq i \leq 2n$, let $i' = 2n + 1 - i$ and $|i| = \min\{|i, i'\}$. Let $w, \tau \in W$, with $w \succeq \tau$. Let $\tau = (a_1 \cdots a_{2n})$. Then the tangent space $T(w, \tau)$ to $X(w)$ at $e(\tau)$ is spanned by the set of root vectors $\{X_{-\beta}, \beta \in N(w, \tau)\}$, where $N(w, \tau)$ is the subset of $\tau(R^+)$ consisting of roots $\beta$ which satisfy criteria (a) and (b) below. Let $\beta = \tau(\alpha), \alpha \in R^+$. We follow the notation of [2] for elements of $R^+$.

(a) Let $\alpha = \varepsilon_j - \varepsilon_k, 1 \leq j < k \leq n$ or $2\varepsilon_j, 1 \leq j \leq n$. Then
\[ w \succeq s_\beta \tau. \]
(b) Let \( \alpha = \varepsilon_j + \varepsilon_k, 1 \leq j < k \leq n \). Let \( s \) (resp. \( r \)) be the \( \min \{|a_j|, |a_k|\} \) (resp. \( \max \{|a_j|, |a_k|\} \)). Then

\[
\omega^{(j)} \geq (a_1, \ldots, a_{j-1}, a'_k)
\]

and

\[
\omega^{(k)} \geq (a_1, \ldots, a_j, \ldots, a_{k-1}, r, s')
\]

II. The special orthogonal group \( \text{So}(2n + 1) \). Let

\[
E = \begin{pmatrix}
0 & 1 \\
\vdots & \ddots \\
1 & 0
\end{pmatrix}_{2n+1 \times 2n+1}
\]

and let \(( , )\) be the symmetric bilinear form on \( k^{2n+1} \), represented by \( E \), with respect to \( \{e_1, \ldots, e_{2n+1}\} \). Let

\[
G = \text{So}(2n + 1) = \{ A \in \text{SL}(2n + 1) | ^tAEA = E \}.
\]

Let \( \sigma \) be the involution on \( \text{SL}(2n + 1) \) defined by

\[
\sigma(A) = E(^tA)^{-1}E, \quad A \in \text{SL}(2n + 1).
\]

As in §I, we have

\[
\text{So}(2n + 1) = \text{SL}(2n + 1)^\sigma.
\]

In view of (15), we obtain identifications for the Weyl group \( W \), and also for \( W_{P^d} \) similar to (9) and (10), namely

\[
W = \{(a_1 \cdots a_{2n+1}) \in S_{2n+1} | a_i = 2n + 2 - a_{2n+2-i}, 1 \leq i \leq 2n + 1 \}
\]

and

\[
W_{P^d} = \left\{ (a_1, \ldots, a_d) \middle| \begin{array}{l}
(1) \ 1 \leq a_1 < a_2 < \cdots < a_d \leq 2n + 1, \\
(2) \ a_i \neq n + 1, 1 \leq i \leq d, \\
(3) \text{For } 1 \leq i \leq 2n + 1, \text{ if } i \in \{a_1, \ldots, a_d\}, \text{ then } 2n + 2 - i \notin \{a_1, \ldots, a_d\}
\end{array} \right\}.
\]

For \( w \in W \), say \( w = (c_1 \cdots c_{2n+1}) \), we have

\[
w^{(d)} = (c_1, \ldots, c_d) \uparrow.
\]

As in §I, we have (cf. [10]) that the Bruhat order in \( W_{P^d} \) coincides with the natural order (cf. equation (3)) on \( d \)-tuples.

**Proposition B.1.** (Assume \( \text{char } k \neq 2 \).) Let \( G = \text{So}(2n + 1) \). For \( 1 \leq i \leq 2n + 1 \), let \( i' = 2n + 2 - i \) and \( |i| = \min\{i, i'\} \). Let \( w, \tau \in W \), with \( w \geq \tau \), and let \( \tau = (a_1 \cdots a_{2n+1}) \). Then the tangent space \( T(w, \tau) \) to \( X(w) \) at \( e(\tau) \) is spanned by the set of root vectors \( \{X_\beta, \beta \in N(w, \tau)\} \), where \( N(w, \tau) \) is the subset of \( \tau(R^+) \) consisting of roots \( \beta \) which satisfy criteria (a), (b), and (c) below. Let \( \beta = \tau(\alpha), \alpha \in R^+ \).

(a) Let \( \alpha = \varepsilon_j - \varepsilon_k, 1 \leq j < k \leq n \). Then

\[
w \geq s_{\beta \tau}.
\]
(b) Let $\alpha = \epsilon_j + \epsilon_k$, $1 \leq j < k \leq n$. Let $s$ (resp. $r$) be the minimum (resp. maximum) of $\{|a_j|, |a_k|\}$.

(i) Suppose precisely one of $\{a_j, a_k\}$ does not exceed $n$. Then

$$w^{(j)} \succeq (a_1, \ldots, a_{j-1}, a'_k) \uparrow,$$

$$w^{(k)} \succeq (a_1, \ldots, a_j, \ldots, a_{k-1}, r, s') \uparrow,$$

and

$$w^{(n)} \succeq (s\varphi)^{(n)}.$$

(ii) Suppose $a_j, a_k$ either both exceed $n$ or both do not exceed $n$. For $k < d < n - 1$, let $s_{c(d)}$ be the largest integer, $r < s_{c(d)} \leq n$, such that $s_{c(d)} \notin \{|a_1|, \ldots, |a_d|\}$ (if no such integer exists, we let $s_{c(d)} = r$). Then

$$w^{(j)} \succeq (a_1, \ldots, a_{j-1}, a'_k) \uparrow,$$

$$w^{(d)} \succeq (a_1, \ldots, a_j, \ldots, a_k, \ldots, a_d, s'_{c(d)}, s') \uparrow, \quad k \leq d \leq n - 1,$$

and

$$w^{(n)} \succeq (s\varphi)^{(n)}.$$

(c) Let $\alpha = \epsilon_j$, $1 \leq j \leq n$. For $j \leq d \leq n - 1$, let $s_{m(d)}$ be the largest integer, $|a_j| < s_{m(d)} \leq n$, such that $s_{m(d)} \notin \{|a_1|, \ldots, |a_d|\}$ (if no such $s_{m(d)}$ exists, we let $s_{m(d)} = |a_j|$). Then

$$w^{(d)} \succeq (a_1, \ldots, a_j, \ldots, a_d, s'_{m(d)}) \uparrow, \quad j \leq d \leq n - 1,$$

and

$$w^{(n)} \succeq (s\varphi)^{(n)}.$$

III. The special orthogonal group $\text{So}(2n)$. Let

$$E = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}_{2n \times 2n},$$

and let $(\ , \ )$ be the symmetric bilinear form on $k^{2n}$, represented by $E$, with respect to $\{e_1, \ldots, e_{2n}\}$. Let

$$G = \text{So}(2n) = \{A \in \text{SL}(2n) | \ tAEA = E\}.$$ 

Let $\sigma$ be the involution on $\text{SL}(2n)$ defined by

$$\sigma(A) = E(tA)^{-1}E, \quad A \in \text{SL}(2n).$$

We have

$$\text{So}(2n) = \text{SL}(2n)^\sigma.$$ 

As in §§I and II, we obtain, in view of (21), identifications (described below) for $W$ and $W^P_\sigma$. We have

$$(22) \quad W = \left\{(a_1 \cdots a_{2n}) \in S_{2n} \middle| \begin{array}{l} (1) \ a_i = 2n + 1 - a_{2n+1-i}, \ 1 \leq i \leq 2n, \\ (2) \ #{\{i, \ 1 \leq i \leq n| \ a_i > n\} \ is \ even} \end{array} \right\}.$$
For $1 \leq d \leq n$, let

$$Z_d = \left\{ (a_1, \ldots, a_d) \mid \begin{array}{l} (1) \quad 1 \leq a_1 < a_2 < \cdots < a_d \leq 2n, \\
(2) \quad \text{for } 1 \leq i \leq 2n, \text{ if } i \in \{a_1, \ldots, a_d\}, \text{ then } 2n + 1 - i \notin \{a_1, \ldots, a_d\} \end{array} \right\}. $$

We have for $d \neq n - 1$

$$W^d = Z_d. $$

For $d = n - 1$, if $w \in W^d$, then

$$w \equiv w_{u_i} \pmod{W_{P_{n-1}}}, \quad 0 \leq i \leq n, \quad i \neq n - 1,$$

where

$$u_i = \begin{cases} s_{\alpha_i} & \text{if } i = n, \\
\text{Id} & \text{if } i = 0, \\
s_{\alpha_i} s_{\alpha_{i+1}} \cdots s_{\alpha_n} \text{ if } 1 \leq i \leq n - 2. \end{cases} $$

(Here $\text{Id}$ denotes the identity element in $W$.) In particular, for $w_1, w_2 \in W$, say $w_1 = (a_1 \cdots a_{2n})$, $w_2 = (b_1 \cdots b_{2n})$, we can have $w_1^{(n-1)} = w_2^{(n-1)}$ without $(a_1, \ldots, a_{n-1}) \uparrow$ and $(b_1, \ldots, b_{n-1}) \uparrow$ being the same. Thus $W_{P_{n-1}}$ gets identified with a proper subset of $Z_{n-1}$ (cf. Definition (23)). For $w \in W$, say $w = (c_1 \cdots c_{2n})$, we have

$$w^{(d)} = (c_1, \ldots, c_d) \uparrow, \quad d \neq n - 1. $$

To describe $w^{(n-1)}$, we let, for $1 \leq i \leq n, \ i \neq n - 1$,

$$y^{(i)}_1, \ldots, y^{(i)}_{n-1} = \left\{ \begin{array}{l} \text{the } (n-1)-\text{tuple given by the first } (n-1) \\
\text{entries in } w_{u_i} \end{array} \right\}$$

and

$$Y = \{(y^{(i)}_1, \ldots, y^{(i)}_{n-1}) \uparrow, \ 0 \leq i \leq n, \ i \neq n - 1\}. $$

We observe that $Y$ is totally ordered under $\geq$ (cf. (3)). We have

$$w^{(n-1)} = \text{the smallest (under } \geq) \text{ element in } Y. $$

Unlike the cases of $\text{Sp}(2n)$ (resp. $\text{So}(2n + 1)$), the Bruhat order in $W$, the Weyl group of $\text{So}(2n)$, is not induced from the Bruhat order in $S_{2n}$. Hence the Bruhat order in $W^d$ does not coincide with the natural order on $d$-tuples (cf. (3)). We now describe the Bruhat order in $W^d$. For $1 \leq i \leq 2n, \ i'$

$$i' = 2n + 1 - i \quad \text{and } |i| = \min\{i, i'\}. $$

Under the above identification, given two elements $(a_1, \ldots, a_d), (b_1, \ldots, b_d)$ in $W^d, 1 \leq d \leq n$, we have (cf. [10])

$$(a_1, \ldots, a_d) \geq (b_1, \ldots, b_d)$$

if and only if the following two conditions hold:

(A) $(a_1, \ldots, a_d) \geq (b_1, \ldots, b_d).$
(B) Suppose for some \( r, 1 \leq r \leq d \), and some \( i, 0 \leq i \leq d - r \),

\[
(|a_{i+1}|, \ldots, |a_{i+r}|) \uparrow = (|b_{i+1}|, \ldots, |b_{i+r}|) \uparrow = \{n + 1 - r, \ldots, n\}.
\]

Then

\[
\#\{j, \ i + 1 \leq j \leq i + r \mid a_j > n\}
\]

and

\[
\#\{j, \ i + 1 \leq j \leq i + r \mid b_j > n\}
\]

should both be odd or both even.

**Proposition D.1.** (Assume \( \text{char } k \neq 2, 3 \).) Let \( G = \text{SO}(2n) \). Let \( w, \tau \in W \), with \( w \succeq \tau \), and let \( \tau = (a_1 \cdots a_{2n}) \). Then the tangent space \( T(w, \tau) \) to \( X(\omega) \) at \( e(\tau) \) is spanned by the set of root vectors \( \{X_\beta, \beta \in N(w, \tau)\} \), where \( N(w, \tau) \) is the subset of \( \tau(R^+) \) consisting of roots \( \beta \) which satisfy criteria (a) and (b) below. Let \( \beta = \tau(\alpha), \alpha \in R^+ \).

(a) Let \( \alpha = \varepsilon_j - \varepsilon_k, 1 \leq j < k \leq n \). Then

\[
w \succeq s_{\beta \tau}.
\]

(b) Let \( \alpha = \varepsilon_j + \varepsilon_k, 1 \leq j < k \leq n \). Let \( s \) (resp. \( r \)) be the minimum (resp. maximum) of \( \{|a_j|, |a_k|\} \).

(i) Suppose precisely one of \( \{a_j, a_k\} \) does not exceed \( n \). Then

\[
w^{(j)} \succeq (a_1, \ldots, a_{j-1}, a'_k) \uparrow,
\]

\[
w^{(k)} \succeq (a_1, \ldots, a_j, \ldots, a_{k-1}, r, s') \uparrow,
\]

\[
w^{(n-1)} \succeq (s_{\beta \tau})(n-1),
\]

and

\[
w^{(n)} \succeq (s_{\beta \tau})(n).
\]

(ii) Suppose \( a_j, a_k \) either both exceed \( n \) or both do not exceed \( n \). For \( k \leq d \leq n - 2 \), let \( s_{-l(d)}, s_{-l(d)-1}, s_0, s_1, \ldots, s_{c(d)} \) be the integers

\[
s < s_{-l(d)} < s_{-l(d)+1} < \cdots < s_{-1} < s_0 = r < s_1 < \cdots < s_{c(d)} \leq n
\]

such that

\[
s_i \notin \{|a_1|, \ldots, |a_d|\}, \quad -l(d) \leq i \leq c(d), \quad i \neq 0.
\]

Then

\[
w^{(j)} \succeq (a_1, \ldots, a_{j-1}, a'_k) \uparrow,
\]

\[
w^{(d)} \succeq \begin{cases} (a_1, \ldots, a_j, \ldots, a_d, s'_{c(d)-1}, s') \uparrow & \text{if } (l(d), c(d)) \neq (0, 0), \\ (a_1, \ldots, a_j, \ldots, a_d, r', s') \uparrow & \text{if } (l(d), c(d)) = (0, 0), \end{cases}
\]

and for \( d = n - 1 \) or \( n \),

\[
w^{(d)} \succeq (s_{\beta \tau})(d).
\]
IV. Concluding remarks.

**COROLLARY.** Let \( G \) be of type \( B_n, C_n, \) or \( D_n \) and let \( w \in W. \) Then \( X(w) \) is smooth if and only if \( \#N(w, \text{Id}) = l(w), \) where \( N(w, \text{Id}) \) is given by Proposition C.1, B.1, or D.1 according as \( G \) is of type \( C_n, B_n, \) or \( D_n, \) with \( \tau = \text{Id}, \) the identity element of \( W. \)

**REMARK 1.** For \( G \) of type \( A_n, \) similar results as above are described in [9].

**REMARK 2.** Even if \( \text{char } k = 2 \) or 3 (in the case of special orthogonal groups), using the explicit computations of \( X^{-\beta}Q(\tau, \tau), \) one can still describe \( T(w, \tau) \) in a way similar to Propositions B.1 and D.1.

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