ARGUESIAN LATTICES WHICH ARE NOT LINEAR

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ABSTRACT. A linear lattice is one representable by commuting equivalence relations. We construct a sequence of finite lattices \( A_n \) \((n \geq 3)\) with the properties: (i) \( A_n \) is not linear, (ii) every proper sublattice of \( A_n \) is linear, and (iii) any set of generators for \( A_n \) has at least \( n \) elements. In particular, \( A_n \) is then Arguesian for \( n \geq 7 \). This settles a question raised in 1953 by Jónsson.

1. Introduction. A lattice \( L \) is linear if it is representable by commuting equivalence relations. Jónsson [6] showed that any such lattice is Arguesian. Numerous equivalent forms of the Arguesian law are now known; it is a strong condition with important applications in coordinatization theory [1, 2]. Nevertheless, the question raised by Jónsson, whether every Arguesian lattice is linear, has remained open until now.

Here we describe an infinite family \( \{A_n\} \) \((n \geq 3)\) of nonlinear lattices, Arguesian for \( n \geq 7 \) (and possibly for \( n \geq 4)\), settling Jónsson's question in the negative. Actually, we obtain more: a specific infinite sequence of identities strictly between Arguesian and linear, and a proof that the universal Horn theory of linear lattices is not finitely based.

2. The lattices \( A_n \). Let \( n \geq 3 \). In what follows, all indices are modulo \( n \), i.e., \( x_{i+1} \) means \( x_0 \) when \( i = n - 1 \), etc. Let \( L_n \) be the lattice of all subspaces of a vector space \( v \) \((\dim v = 2n)\) over a prime field \( K \) with at least 3 elements. Let \( \{\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_{n-1}\} \) be a basis of \( v \). Let

\[
(1) \quad m = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle, \quad q_i = \langle \{\alpha_j | j \neq i\} \rangle, \quad p_i = q_i \land q_{i+1},
\]

\[
q_i = \langle \{\alpha_j \rangle, \quad r_i = m \lor \langle \beta_i \rangle, \quad s_i = r_{i-1} \lor r_i,
\]

where \( \langle \cdots \rangle \) denotes linear span. Let

\[
(2) \quad \tilde{A}_n = [0, m] \cup [m, v] \cup \bigcup_i [p_i, r_i] \cup \bigcup_i [q_i, s_i],
\]

where \( [x, y] = \{z | x \leq z \leq y\} \).

\( \tilde{A}_n \subseteq L_n \) is a sublattice; the intervals in the union (2) are its maximal complemented intervals, or blocks; they are the blocks of a tolerance relation on \( \tilde{A}_n \) [5]; as such, the set \( S \) of blocks acquires a lattice structure; specifically, \( 0_S = [0, m], 1_S = [m, v], a_i = [p_i, r_i] \) are atoms, \( b_i = [q_i, s_i] \) are coatoms, and \( a_i < b_i, b_{i+1} \) defines the order relation.

Let \( \overline{m} \) \((\dim \overline{m} = n)\) be another vector space, with basis \( \{\overline{\alpha}_0, \ldots, \overline{\alpha}_{n-1}\} \). Define \( \overline{p}_i, \overline{q}_i \) by analogy with (1). Let \( F = \bigcup_i [p_i, v]; F \subset \tilde{A}_n \) is an order filter. Within \( F, \bigcup_i [p_i, m] \) is an order ideal. Set up a "twisting" isomorphism

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τ of \( \bigcup_i [p_i, m] \) with the order filter \( \bigcup_i [\bar{p}_i, \bar{m}] \subset [0, \bar{m}] \) as follows: for each \( i \), the atoms of \( [\bar{p}_i, \bar{m}] \) are of the form \( \langle r\bar{a}_i + s\bar{a}_{i+1}, \bar{a}_{i+2}, \ldots, \bar{a}_{i-1} \rangle \) where \( (r : s) \) is a ratio of elements of \( K \). Put \( \tau(\bar{p}_i) = p_i \), \( \tau(\bar{m}) = m \), and \( \tau(\langle r\bar{a}_i + s\bar{a}_{i+1}, \bar{a}_{i+2}, \ldots, \bar{a}_{i-1} \rangle) = \langle r\alpha_i + s\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_{i-1} \rangle \) except, when \( i = 0 \), put

\[
\tau(\langle r\bar{a}_0 + s\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n-1} \rangle) = \langle -r\alpha_0 + s\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \rangle.
\]

This definition is consistent on \( \bar{q}_i \) and makes \( \tau(\bar{q}_i) = q_i \).

Let

\[
A_n = F \cup [0, \bar{m}] / (x = \tau(x))_{x \in \bigcup_i [\bar{p}_i, \bar{m}]}.\]

\( A_n \) is a modular lattice and has the same block decomposition (2) as \( \tilde{A}_n \), hence the same skeleton lattice \( \tilde{S} \). Composing \( \tau \) with the automorphism of \( [0, \bar{m}] \) induced by the linear transformation \( \alpha_1 \mapsto -\alpha_1, \ldots, \alpha_k \mapsto -\bar{\alpha}_k \), other \( \alpha_i \) fixed, shows that the exceptional interval \( [\bar{p}_0, \bar{m}] \) in the definition of \( \tau \) could as well have been \( [\bar{p}_k, \bar{m}] \), up to an isomorphism of \( A_n \) respecting the \( \bar{p}^j \).

3. Properties of \( A_n \).

**Theorem.** \( A_n \) is not a linear lattice.

**Proof.** In [3], the author introduced “higher Arguesian identities”

\[
D_n: \quad a_0 \land \left( a'_0 \lor \bigwedge_{i=1}^{n-1} [a_i \lor a'_i]\right) \leq a_1 \lor \left( (a'_0 \lor a'_1) \land \bigvee_{i=1}^{n-1} [(a_i \lor a_{i+1}) \land (a'_i \lor a'_{i+1})]\right)
\]

which hold in all linear lattices. \( D_3 \) is the Arguesian law [4]. If we take \( a_i = p_i + (\beta_i) \) for all \( i \), \( a'_i = p_i + (\beta_i + \alpha_i + \alpha_{i+1}) \) for \( i \neq 0 \), and \( a'_0 = p_0 + (\beta_0 - \alpha_0 + \alpha_1) \), \( D_n \) fails in \( A_n \). In particular, \( A_3 \) is not Arguesian. This minimally non-Arguesian lattice was discovered by Pickering [8].

**Theorem.** Every proper sublattice of \( A_n \) is linear.

**Proof.** \( \bigcup_i [p_i, r_i] \) generates \( A_n \), so a proper sublattice \( N \subset A_n \) will have \( N \cap [p_i, r_i] \subset [p_i, r_i] \) strictly for some \( i \). We can assume \( [p_i, m] \) is the exceptional interval in the definition of \( \tau \). We show \( [p_i, r_i] \) (which is a projective plane over \( K \)) possesses an automorphism fixing \( N \cap [q_i, r_i] \) and \( N \cap [q_i+1, r_i] \) and acting as \( \tau \) on \( N \cap [p_i, m] \). This is proved by classifying maximal proper sublattices of \( [p_i, r_i] \) and their possible orientations relative to \( m, q_i, q_i+1 \), which leads to 13 cases to check, some trivial, none difficult.

It follows that \( A_n \) has a sublattice isomorphic to \( N \), so \( N \) is linear.

**Theorem.** If \( X \subseteq A_n \) generates \( A_n \), then \( |X| \geq n \).

**Proof.** For each \( j \), \( 0_S \cup 1_S \cup \bigcup_{i \neq j} a_i \cup \bigcup_{i \neq j} b_i \) is a sublattice of \( A_n \) because \( \{0_S, 1_S\} \cup \{a_j, b_i | i \neq j\} \) is a sublattice of \( S \). For each \( j \), therefore, some \( x_j \in X \) is an element of \( a_j \cup b_j \) and not an element of any other block. This requires \( n \) distinct elements of \( X \).
4. Conclusions. The results of §3 imply that no finite set of identities, or even universal Horn sentences, can completely characterize linearity; in particular, the Arguesian law is insufficient, since it holds in $A_n$ for $n \geq 7$. It is known, however, how to characterize linear lattices by an infinite set of universal Horn sentences [3, 7].

If, as appears likely, the identity $D_{n-1}$ holds in $A_n$ ($n \geq 4$), we would have that $D_{n-1}$ does not imply $D_n$, showing that $\{D_n\}$ forms a hierarchy of progressively strictly stronger linear lattice identities. We remark that generator-counting will not suffice for this, since $A_n$ has a set of generators $X$ with $|X| = n + 3$. We conjecture $n + 3$ is minimal, which would imply $A_4$ is Arguesian.

REFERENCES


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