A NOTE ON CALDERÓN-ZYGMUND SINGULAR INTEGRAL CONVOLUTION OPERATORS

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The purpose of this note is to show that the notion of weak maximal function introduced in [1] (see also [4], where a similar notion is considered) can be used to obtain some new information on the Calderón-Zygmund singular integral convolution operator.

We will follow the notations of [3]. Let $K$ be a kernel in $\mathbb{R}^n$ of class $C^1$ outside the origin satisfying

1. $|K(x)| \leq C|x|^{-n},$
2. $|\nabla K(x)| \leq C|x|^{-n-1}.$

For $\varepsilon > 0$ and $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$, set

$$T_\varepsilon(f)(x) = \int_{|y| \geq \varepsilon} f(x-y)K(y) \, dy$$

and

$$T(f)(x) = \lim_{\varepsilon \to 0} T_\varepsilon(f)(x), \quad T^*(f)(x) = \sup_{\varepsilon > 0} |T_\varepsilon(f)(x)|.$$

We will assume that $K$ satisfies the usual properties ensuring that the mapping $f \mapsto T^*(f)$ is of weak type $(1,1)$ and that $T(f)(x)$ makes sense for a.e. $x$.

The notation $L^{1,\infty}$ will stand for the space of weak $L^1$ functions, and if $\varphi \in L^{1,\infty}$ and $B$ is a ball we write

$$||\varphi||_{1,\infty}^B = \sup_{\delta > 0} \delta m(\{x \in B: |\varphi(x)| > \delta\})$$

for the weak $L^1$ "norm" of $\varphi$ on $B$. If $B = \mathbb{R}^n$, we simply write $||\varphi||_{1,\infty}$.

The weak maximal function introduced in [1] is defined for $\varphi \in L^{1,\infty}$ by

$$M_w \varphi(x) = \sup_{x \in B} \frac{||\varphi||_{1,\infty}^B}{m(B)},$$

the supremum being taken over all balls centered at $x$. The notation $M_w^m \varphi$ stands for the function obtained by applying $m$ times the operator $M_w$, whenever this makes sense. In [1] it was already pointed out that for any $m$ there is a $\varphi \in L^{1,\infty}$ such that $M_w^j \varphi \in L^{1,\infty}$ for $j = 1, \ldots, m$ but $M_w^{m+1} \varphi \notin L^{1,\infty}$.

However, for $\varphi = T f$, $f \in L^1$, the following holds:

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THEOREM. If $T$ is as above, $M_w^m T^* f \in L^{1,\infty}$ for all $f \in L^1$ and all $m \in \mathbb{N}$, and $||M_w^m T^* f||_{1,\infty}$ grows as a geometric progression. Hence the same is true for $T f$.

As in [1], the motivation for this research is the following QUESTION. What is the necessary and sufficient condition on a nonnegative function $\varphi$ for the existence of $f \in L^1$ such that

$$\varphi \leq |Tf| \quad \text{a.e.}$$

In other words, what is the precise description of the magnitude of $Tf$? The theorem gives a necessary condition stronger than $\mathcal{L}^1,0^\circ$, namely,

$$||M_w^m \varphi||_{1,\infty} \leq C_1 C_2^m$$

but, as shown in the last section of [1], this condition is not sufficient (see §5 of [1] for other remarks concerning this question).

PROOF OF THE THEOREM. Let us first remark that the corresponding result with $Tf$ replaced by the Hardy-Littlewood maximal function $Mf$ is also true. In fact in this case something more precise is true, namely

(3) $M_w Mf(x) \leq C Mf(x)$

for some constant $C = C(n)$. This is shown in [1, pp. 9-10], and it is also implicit in [2].

In fact, our proof of the theorem will be a consequence of something similar to (3). We will show that

(4) $M_w T^* f \leq C \{T^* f + Mf\}.$

Together with (3) this will give

$$M_w^m T^* f \leq C^m \{T^* f + Mf\}$$

(note that $M_w (\varphi + \psi) \leq 2 (M_w \varphi + M_w \psi)$), which clearly implies the theorem.

In order to prove (4), fix $x$ and let $B$ be a ball centered at $x$. Let $2B$ denote the ball having the same center as $x$ and twice the radius and set

$$f_1 = f |_{x \in 2B}, \quad f_2 = f - f_1.$$

Then $T^* f \leq T^* f_1 + T^* f_2$ and

$$||T^* f||^B_{1,\infty} \leq 2 (||T^* f_1||^B_{1,\infty} + ||T^* f_2||^B_{1,\infty}).$$

Since $T^*$ satisfies a weak $(1,1)$ estimate, we have

(5) $||T^* f_1||^B_{1,\infty} \leq ||T^* f_1||_{1,\infty}$

$$\leq C ||f_1||_1 = C \int_{2B} |f(y)| \, dy \leq Cm(B) Mf(x).$$

Now we will prove that for $z \in B$

(6) $||T^* f_2(z)|| \leq C (T^* f(x) + Mf(x)).$

This implies

$$||T^* f_2||^B_{1,\infty} \leq Cm(B) (T^* f(x) + Mf(x)),$$
and together with (5) this gives
$$\|T^* f\|_{1,\infty}^B \leq Cm(B)(T^* f(x) + Mf(x)),$$
which is (4).

For (6) we have to prove that for any $\epsilon > r$, $r$ being the radius of $B$,
$$|T_\epsilon f_2(z)| \leq C(T^* f(x) + Mf(x)).$$
Now
$$T_\epsilon f_2(z) = \int_{R^n \setminus 2B} f(y)K(z - y) \, dy.$$
If $\delta = \epsilon + r$, it is clear that the contribution in this integral of $B_0 = \delta B/r$, the ball centered at $x$ of radius $\delta$, is dominated (using (1)) by
$$C\epsilon^{-n} \int_{B_0} |f(y)| \, dy \leq CMf(x).$$

It remains to estimate
$$I \overset{\text{def}}{=} \int_{|y - z| > \delta} f(y)K(z - y) \, dy.$$
This is compared with $T_\delta f(x)$ in the usual way:
$$I - T_\delta f(x) = \int_{|y - z| > \delta} f(y)\{K(z - y) - K(x - y)\} \, dy.$$
Using (2) we obtain
$$|I - T_\delta f(z)| \leq C|z - x| \int_{|y - z| > \delta} |f(y)| \frac{dy}{|y - x|^{n+1}},$$
and it is well known that this is in turn dominated by $Mf(x)$. Therefore we have proved that
$$|T_\epsilon f_2(z)| \leq C(|T_\delta f(x)| + Mf(x)),$$
which yields (7) and finishes the proof of the theorem.

REMARK. S. Drury (private communication) has independently generalized some results of [1] proving that $MwTf \in L^{1,\infty}$, i.e. the case $m = 1$ of the Theorem. He replaces the condition (1) and (2) by the so-called Hörmander condition (see [3, p. 34, condition (2')]).

REFERENCES