AMALGAMATIONS AND THE KERVAIRE PROBLEM

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ABSTRACT. Following S. Brick, a 2-complex $X$ is called “Kervaire” if all systems of equations, with coefficients in arbitrary groups $G$ and the attaching maps of $X$ as the words in the variable letters, are solvable in an overgroup of $G$. An obstruction theory is developed for solving equations modeled on $Z = \frac{X \cup Y}{\Gamma}$, where $X$ and $Y$ are Kervaire 2-complexes and $\Gamma$ is a subgraph of $Z^{(1)}$, each connected component of which injects at the $\pi_1$-level into $\pi_1(Z)$. A 2-complex of the form $K(\Gamma ; w(x) = w'(y))$ is Kervaire, where $w(x)$ and $w'(y)$ are (not necessarily reduced) words which do not freely reduce to 1.

The Kervaire problem [7, p. 403] originally asked whether a nontrivial group can be killed by adjoining a single free generator and a single relator. This problem has been vastly generalized by Howie [5], who asked whether a system of equations over an arbitrary coefficient group $G$, whose words in the variable letters are the attaching maps of a 2-complex $X$ with $H_2(X) = 0$, is solvable in an overgroup of $G$. It is convenient to introduce a terminology due to S. Brick [1] who calls a 2-complex $X$ Kervaire iff all systems of equations over all coefficient groups $G$ modeled on the attaching maps of $X$ are solvable in an overgroup of $G$. Thus, e.g., the dunce hat $K(x; xx\bar{x})$ is Kervaire because Howie has shown that the equation $axbxc\bar{x} = 1$, with $a, b, c \in G$, can always be solved in an overgroup of $G$ [6].

In this terminology, a nontrivial group can never be killed by adjoining a single free generator and a single relator if the 2-complex $K(x; w(x))$ is Kervaire, where $w(x)$ is a word in $x$ and $x^{-1}$ whose exponent sum in $x$ is $\pm 1$.

For a 2-complex with one 2-cell $X = K(x_1, x_2, \ldots, x_n; w(\bar{x}))$ Howie’s problem can be shown (nontrivially) to imply that $X$ is Kervaire iff $w(\bar{x})$ does not freely reduce to 1 (the “if” assertion is the nontrivial one here). Since $X = K(\bar{x}; w(\bar{x}))$ can be easily shown to be Cockcroft iff $w(\bar{x})$ does not freely reduce to 1, Howie’s problem for 2-complexes $X$ with one 2-cell amounts to
the assertion that \( X \) is Kervaire iff \( X \) is Cockcroft (recall a 2-complex \( X \) is Cockcroft iff the Hurewicz homomorphism \( \pi_2(X) \to H_2(X) \) is zero).

We can prove

**Theorem 1.** Let \( x_1^\pm, x_2^\pm, \ldots, x_n^\pm \) and \( y_1^\pm, \ldots, y_m^\pm \) be disjoint alphabets and let \( w(\bar{x}) \) and \( w'(\bar{y}) \) be words in these alphabets respectively which do not freely reduce to 1. Then \( K(x_1, \ldots, x_n, y_1, \ldots, y_m | w(\bar{x}) = w'(\bar{y})) \) is Kervaire.

This result can be stated in the equivalent form below, more appealing to topologists, by recalling the connected sum \( X \# Y \) of two 2-complexes [8]. One chooses imbeddings of the disc \( D^2 \) in \( X \) and \( Y \) respectively, each with one point contact with \( X \) (and \( Y \)) one bores out the interiors of the discs, and one identifies their boundaries to get \( X \# Y \). The construction depends sensitively on the choice of imbeddings of discs.

**Theorem 2.** Let \( X \) and \( Y \) be Cockcroft 2-complexes each possessing only one 2-cell. Then \( X \# Y \) is Kervaire (for all choices of imbedded discs in \( X \) and \( Y \) as above).

The main technical innovation is an obstruction theory for deciding when \( Z = X_1 \cup X_2 \) is Kervaire provided \( \Gamma \) is a subgraph of \( Z \) such that \( \pi_1 \) of each connected component of \( \Gamma \) injects into \( \pi_1(Z) \) (S. Brick calls such an inclusion \( \Gamma \to Z \) \( \pi_1 \)-injective [1]). Let \( f : (D^2, S^1) \to (X, \Gamma) \) be a combinatorial map (for some cell structure on \( D^2 \)). We define the obstruction element \( \Lambda(f) \in G_f \ast (E(\Gamma)) \) to be the product in order of corner labels and edge labels in one full circuit around \( \partial D^2 \); here \( G_f \) is the factor group of the corner group [4] of \( X \) modulo interior vertex labels of \( f \) and \( (E(\Gamma)) \) denotes a free group freely generated by an oriented set of edges of \( \Gamma \). The technical result is the following

**Theorem 3.** Let \( Z = X_1 \cup X_2 \), where the inclusion \( \Gamma \to Z \) is \( \pi_1 \)-injective. Assume that \( X_1 \) and \( X_2 \) are Kervaire and that all obstruction elements \( \Lambda(f) = 1 \) for all maps \( (D^2, S^1) \to (X_i, \Gamma), i = 1, 2 \). Then \( Z \) is Kervaire.

An example where all obstructions \( \Lambda(f) \) vanish is where \( \Gamma \) is 2-sided in \( Z \).* In this case Theorem 3 implies as a corollary a result of Brick's thesis [1]: if \( \Gamma \) is a subgraph of \( Z \) such that the inclusion \( \Gamma \to Z \) is \( \pi_1 \)-injective and \( \Gamma \) is 2-sided in \( Z \) and if in addition the result of cutting \( Z \) along \( \Gamma \) is Kervaire, then \( Z \) is Kervaire.

To apply Theorem 3 we need to calculate obstructions. Let \( X = K(x_1, \ldots, x_n, t | t = w(\bar{x})) \) and let \( \Gamma = K(t) \), a subgraph of \( X \). The inclusion \( \Gamma \to X \) is \( \pi_1 \)-injective iff the word \( w(\bar{x}) \in F(\bar{x}) \) does not freely reduce to 1. We prove

**Theorem 4.** For any combinatorial map \( f : (D^2, S^1) \to (X, \Gamma) \), where \( X \) and \( \Gamma \) are as defined immediately above and where \( w(\bar{x}) \) does not freely reduce to 1, one has \( \Lambda(f) = 1 \).

* \( \Gamma \) is called "2-sided" in \( Z \) if it is bicolliared: so \( \Gamma \) is identified with \( \Gamma \times \{1/2\} \) where \( \Gamma \times [0, 1] \) is a product neighborhood of \( \Gamma \) in \( Z \).
The proof of Theorem 4 proceeds by assuming \( f \) is reduced (so no two 2-cells of \( D^2 \) with an edge \( e \) in common are mapped mirror-wise across \( e \)) and showing that, by small cancellation type arguments, in this reduced case the domain has a vertex of valence 1 in its 1-skeleton. This enables us to do 2-bridge moves and at the same time reduce the size of \( w(x) \) by cancelling an adjacent pair of cancelling letters. The argument proceeds by an induction on the length of \( w(\overline{x}) \), the induction beginning when \( w \) is a reduced word (\( \neq 1 \)); in this case one sees directly no such reduced maps \( f \) can exist.

Theorem 2 follows from Theorems 3 and 4 by appealing to the subdivision theorem for Kervaire complexes [1] and by observing that the complex \( X \) in Theorem 4 collapses onto a graph and is hence Kervaire.

Similar arguments establish the following result. Recall that a 2-complex \( X \) is called diagrammatically reducible (DR) [4] if there are no reduced combinatorial maps of \( S^2 \) to \( X \).

**THEOREM 5.** Let \( w_i(\overline{x}), i \in I \), be a set of words in the alphabet \( \overline{x} = (x^y_1, \ldots, x^y_n) \) and assume that the elements in the free group \( F(\overline{x}) \) these words \( w_i(\overline{x}) \) represent freely generate the subgroup \( S \) of \( F(\overline{x}) \). If no proper initial segment of any word \( w_i(\overline{x}) \) represents an element of \( S \), then the 2-complex

\[
K\langle x_1, \ldots, x_n, y_1, \ldots, y_n \mid w_i(\overline{x}) = w_i(\overline{y}), i \in I \rangle
\]

is diagrammatically reducible.

**COROLLARY.** If \( F \) is a free group and \( A \leq F \), then the double of \( F \) along \( A, F *_A F \), has a DR presentation.

It is an open question whether every aspherical 2-complex is homotopy equivalent to a DR 2-complex (see [2, §6] for additional examples, drawn from 3-manifold theory, where this is true).

Theorem 5 above has an amusing illustration. It follows immediately that the presentation \( (x, y, z, w|x^n y^n z^n w^n, \forall n \geq 1) \) is DR. This implies [4] that for any group \( G \) and sequence of elements \( a_n \in G \), \( n \geq 1 \), the system of equations

\[
a_n = x^n y^n z^n w^n, \quad \forall n \geq 1,
\]

can be simultaneously solved in an overgroup of \( G \).

Another explicit calculation of the obstruction element \( \Lambda(f) \) shows there is a 2-complex which is Cockcroft but not Kervaire. Explicitly we have

**THEOREM 6.** Let \( X = K(\{x, y, t\} | x^2, y^2, t = xy) \). Let \( \Gamma = K(t | ) \), a \( \pi_1 \)-injective subgraph of \( X \). Then the double \( Z \) of \( X \) along \( \Gamma \), \( Z = X_{\Gamma} X \), is Cockcroft and diagrammatically aspherical but not Kervaire.

“Diagrammatically aspherical” here means that given any combinatorial map of a cell structure \( S^2 \) to \( Z \), some sequence of diamond moves exists which splits off a component 2-sphere with precisely two faces. The example \( Z \) of Theorem 6 is interesting because the homotopy equivalent 2-complex

\[
W = (X \times (0))_{\Gamma \times (0)}(\Gamma \times I)_{\Gamma \times (1)}(X \times (1))
\]
is Kervaire, as one sees by applying Brick's 2-sided $\pi_1$-injective theorem quoted after Theorem 3. It follows that the property of being Kervaire is not a homotopy type invariant of 2-complexes.

Suppose now that $X = K(P)$, where $P$ is the finite presentation $P = \langle x_1, x_2, \ldots, x_n, t_i \mid i \in I \rangle$, and let $\Gamma = K(t_i(i \in I))$, a subgraph of $X^{(1)}$ (so $X$ collapses cellularly onto a subgraph of $X^{(1)}$ with $E(\Gamma)$ as the set of free edges for the collapse). Let $Z = X \cup_\Gamma X$, the double of $X$ along $\Gamma$. It is easy to see that the inclusion $\Gamma \to X$ is $\pi_1$-injective iff $Z$ is Cockcroft iff $Z$ is aspherical iff $\{w_i(\vec{x}), i \in I\}$ is freely independent in $F(\vec{x})$.

**Theorem 7.** If $Z$ is Kervaire, then the inclusion $\Gamma \to X$ is $\pi_1$-injective. Furthermore if $\Gamma \to X$ is $\pi_1$-injective and we assume either a positive solution to Howie's problem or the invariance of Kervaire complexes (with one vertex) under Andrews-Curtis moves, then $Z$ is Kervaire.

Theorem 5 is used in proving the last assertion in Theorem 7 as follows. If $\{w_i(\vec{x}), i \in I\}$ is independent, then one may do Nielsen moves to transform this collection to a Schreier basis for the subgroup generated; here Theorem 5 applies. On the other hand Nielsen moves on $\{w_i(\vec{x}), i \in I\}$ correspond to Andrews-Curtis moves on $Z$, so invariance of the Kervaire property under these latter moves implies that $Z$ is Kervaire.

In this connection I have developed an algorithm for generating all reduced disc diagrams $f : (D^2, S^1) \to (X, \Gamma)$ with $(X, \Gamma)$ as in Theorem 7. The algorithm is "smart" in the sense that it can select certain diagrams for which $\Lambda(f) = 1$ because of the known positive results about the Howie problem. Hand computations have so far led to no "interesting" diagrams, where a diagram is called "interesting" if these selection rules don't automatically imply $\Lambda(f) = 1$. The algorithm ought to be programmed on a high-speed computer, to continue the search for "interesting" diagrams.

**References**


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