SMOOTH NONTRIVIAL 4-DIMENSIONAL $s$-COBORDISMS

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ABSTRACT. This announcement exhibits smooth 4-dimensional manifold triads $(W; M_0, M_1)$ which are $s$-cobordisms, i.e. the inclusions $M_i \subseteq W$, $i = 0, 1$, are simple homotopy equivalences, but are not diffeomorphic or even homeomorphic to a product $M_i \times [0,1]$.

The Barden-Mazur-Stallings $s$-cobordism theorem constitutes one of the foundational stones of modern topology. It asserts, in the smooth, piecewise-linear, or topological categories, that if $W$ is a manifold of dimension at least six, with boundary components $M_i$, $i = 0, 1$, whose inclusions into $W$ are simple homotopy equivalences, then $W$ is necessarily a product (see [K, H, RS, KS]). For simply connected smooth manifolds of dimension at least six, this result had already been proven by Smale as the "h-cobordism theorem" [Sm], with the generalized Poincaré conjecture in higher dimensions as a corollary. The $s$-cobordism statement holds in dimensions one and two, and is equivalent to the Poincaré conjecture in dimension three. Freedman [F1, F2] proved the five-dimensional result for topological manifolds with fundamental group of polynomial growth (e.g. finite or polycyclic). Donaldson's extraordinary results imply the failure of the five-dimensional result in the smooth (or piecewise linear) category even for simply connected manifolds; by [F1] the resulting $h$-cobordisms will still be topological products. Using Freedman's results, the present authors produced some nontrivial orientable four-dimensional topological $s$-cobordisms [CS1, CS2]. (See [MS] for a nonorientable and definitely nonsmoothable example.) These topological constructions have been further studied and extended by Kwasik and Schultz [KwS].

We will now use a different construction to produce some nontrivial smooth $s$-cobordisms. Neither the construction nor the proof rely on any of the results cited above. Let $M$ be a quaternionic space-form; i.e.

$$M = M_r = S^3/Q_r,$$

$Q_r$ the quaternionic group of order $2^{r+2}$. Then it is well known that the orientable manifold $M$ has a one-sided Heegaard splitting

$$M = N(K) \cup H,$$

where $N(K)$ is the total space of an interval bundle over the Klein bottle $K$ and $H$ is a solid torus. Let $E_0$ be a closed tubular neighborhood of
\[ K = K \times \{0\} \text{ in } M \times (-1,1). \] Then \( E_0 \) is a linear \( D^2 \)-bundle over \( K \) with boundary the double of \( N(K) \). Let

\[ X = M \times [-1,1] - \text{Int } E_0. \]

The smooth \( s \)-cobordisms will be of the form

\[ W = W_r = X \cup_{\partial E_0} E, \]

where \( E \) will be a locally trivial smooth fiber-bundle over \( K \) with fiber \( T^2_0 = S^1 \times S^1 - \text{Int } D^2 \), with \( \partial E = \partial E_0 \).

In fact, view \( S^1 \subset C \) and define \( \psi_i, i = 1,2 \), by

\[ \psi_1(x,y) = (y,yx) \quad \text{and} \quad \psi_2(x,y) = (y^{-1},y^{-1}x^{-1}). \]

Note that \( \psi_1^2 = \psi_2^2 \). The Klein bottle \( K \) is the union of two Möbius bands, and it follows that there is a canonical \( T^2 \)-bundle \( E_1 \) over \( K \) whose restrictions to the cores of the Möbius bands have monodromies \( \psi_1 \) and \( \psi_2 \) respectively. Since \( \psi_1(1,1) = (1,1) \), this bundle has a cross-section, and there is a canonical way to identify a tubular neighborhood of its image with \( E_0 \). We then take \( E = E_1 - \text{Int } E_0 \) in the above definition of \( W \). Clearly, \( W \) is an orientable smooth 4-manifold with two copies of \( M = S^3/Q_r \) as boundary.

**Theorem.** 1. The smooth four-manifold \( W \) is an \( s \)-cobordism of \( M \) to itself.

2. \( W \) is not diffeomorphic or even homeomorphic to a product \( M \times [-1,1] \).

It can also be shown that \( W \) is not homeomorphic to any of the topological \( s \)-cobordisms of \([CS2]\), and the smoothability of any of them remains open.

The proof of 1 uses Van Kampen’s theorem and other well-known arguments in homotopy and simple homotopy theory. However, note that the restriction of a suitable diffeomorphism of \( T^2 \) isotopic to \( \psi_i \) represents a square-root of the monodromy of the figure-eight knot.

We indicate the proof of 2 for the case \( r = 1 \), the quaternion group of order eight. Let \( P \) be obtained from \( W \) by identifying \( M \times \{-1\} \) with \( M \times \{1\} \). Then we explicitly construct a framed 5-manifold \( U \) with the following properties:

1. \( \partial U = P \).
2. There is a retraction \( r: U \to M \) inducing isomorphisms on fundamental groups and homology with \( \mathbb{Z}_2 \) coefficients.
3. If \( U_4 \) and \( U_8 \) are the 4-fold and 8-fold covers of \( U \), respectively, then \( |H_2(U_8)||H_2(U_4)|^{-1} \equiv \pm 7 \) (mod 16).

By contrast, we show that were \( W \) a product and \( U \) as above satisfying 1 and 2, the quotient (of odd integers) in 3 would necessarily be congruent to \( \pm 1 \) (mod 16). Because of the possible choices for \( P \) and \( r \), the proof is somewhat involved. It uses the fact, due independently to J. H. Rubinstein \([R]\) and the present authors, that a diffeomorphism or homeomorphism of \( M \) homotopic to the identity will necessarily be isotopic to it. In the course of the proof, the remaining ambiguity of \([KwS]\) concerning the classification of topological \( s \)-cobordisms of \( M \) to itself is resolved, and a remark in \([CS2]\) is corrected.
It would be interesting to know if the universal covering space of $W$ is diffeomorphic to $S^3 \times [0,1]$. This is similar to the situation for the exotic $\mathbb{RP}^4$ of $[CS3]$, whose covering space is also potentially exotic $[AK]$. It is also of interest to observe that for the case $r = 1$, $W$ can be embedded as a codimension zero submanifold of a smooth homotopy 4-sphere.

REFERENCES


