EXOTIC KNOTTINGS OF SURFACES IN THE 4-SPHERE

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1. The main result.

THEOREM. There exists an infinite series \( S_1, S_2, \ldots \) of smooth submanifolds of \( S^4 \) such that:

1. for any \( i, j \) the pairs \( (S^4, S_i), (S^4, S_j) \) are homeomorphic,
2. for any \( i \neq j \) the pairs \( (S^4, S_i), (S^4, S_j) \) are not diffeomorphic,
3. each \( S_n \) is homeomorphic to the connected sum \( \#_{10} \mathbb{RP}^2 \) of 10 copies of the projective plane,
4. \( \pi_1(S^4\backslash S_n) = \mathbb{Z}_2 \),
5. the normal Euler number (with local coefficients) of \( S_n \) in \( S^4 \) is 16.

Actually we show instead of (1) a slightly stronger result, namely, that there are smooth isomorphisms \( \varphi_i \) of tubular neighborhoods of \( S_i \) and \( S_1 \) which can be extended to homeomorphisms of the exterior. But according to (2) \( \varphi_j^{-1} \circ \varphi_i \) cannot be extended to a diffeomorphism of the exteriors. This is surprising, as there is no analogous result in other dimensions. Let \( N \) be a closed smooth submanifold of a closed manifold \( M \) of dimension \( \neq 4 \). Let \( U \) be a smooth tubular neighborhood. Then there are only finitely many diffeomorphism types rel. boundary of smooth manifolds \( X \) with \( \partial X = \partial U \) and \( X \) homeomorphic to \( M - \bar{U} \). If \( \dim M = 3 \) the number of diffeomorphism types is 1 and if \( \dim M \geq 5 \) the number of smoothings rel. boundary (which is an upper bound for the number of diffeomorphism types) is finite by [KS].

In fact we describe an infinite family \( F_1, F_2, \ldots \) of smooth submanifolds of \( S^4 \) satisfying conditions (2)–(5) of the Theorem, and we prove that there are only finitely many homeomorphism types of \( (S^4, F_n) \) in the sense described above.

The \( F_n \)'s are obtained from a fixed smooth submanifold \( F \subset S^4 \) by a family of new knotting constructions. \( F \) is the obvious simplest submanifold satisfying the conditions (3), (4), and (5): the pair \( (S^4, F) \) is the connected sum of the standard pair \( (S^4, \mathbb{RP}^2) \) (with normal Euler number \(-2\)) and nine copies of it with the orientation of \( S^4 \) reversed.

Our knotting constructions can be applied to “smaller” submanifolds, e.g. the Klein bottle with normal Euler number 0 and the torus, which are standardly embedded in \( S^4 \). The only thing we fail to prove in these situations is the nonexistence of diffeomorphisms.
2. Outline of the proof. The construction of $F_n$ is motivated by the recent work of S. Donaldson [Di, D2] C. Okonek and A. van de Ven [OV] resp. R. Friedman and J. Morgan [FM]. They considered the Dolgachev surfaces [Do], which are complex elliptic surfaces organized in families $D_{p,q}$ with $p, q \in \mathbb{N}$, $(p, q) = 1$. The $D_{p,q}$-surfaces are 1-connected and permit an elliptic fibration over the sphere $CP^1$ with two multiple fibres of multiplicity $p$ and $q$. Any $D_{p,q}$-surface can be obtained from some rational elliptic surface diffeomorphic to $CP^2$ by logarithmic transformations [BPV] of multiplicity $p$ and $q$ along two nonsingular fibres. The rational elliptic surfaces themselves are included in this system as $D_{1,1}$. By Freedman's classification of 1-connected closed 4-manifolds [F1], all Dolgachev surfaces are homeomorphic to $CP^2 \# 9CP$. But Donaldson [Di, D2] proved that no $D_{2,3}$-surface is diffeomorphic to $CP^2 \# 9CP$, and this was extended in [OV, FM], showing that no $D_{r,1}$-surface is diffeomorphic to a $D_{1,1}$-surface or a $D_{2,3}$-surface with odd $r \neq q$.

**Proposition 1.** For any $p, q$ there exists a $D_{p,q}$-surface $M$ which admits an antiholomorphic involution $c$ with $M/c$ diffeomorphic to $S^4$.

In the case of $D_{1,1}$ such an involution can easily be constructed via the usual complex conjugation $c: CP^2 \to CP^2$: $(z_0 : z_1 : z_2) \mapsto (\bar{z}_0 : \bar{z}_1 : \bar{z}_2)$. The orbit space $CP^2/c$ is diffeomorphic to $S^4$ [K, M]. The fixed point set $RP^2$ is standardly embedded in it with normal Euler number $-2$. For any $D_{1,1}$-surface $M$ obtained by blowing up 9 real points (i.e. points in $RP^2$) of $CP^2$, we can extend this involution to an antiholomorphic involution $c$ on $M$ with $M/c$ diffeomorphic to $S^4(= \#_{10}S^4)$. The fixed point set is $F = RP^2 \# 9RP^2 = \#_{10}RP^2 \hookrightarrow S^4$ with normal Euler number $-2 + 9 \cdot 2 = 16$, the standardly embedded $\#_{10}RP^2$ with this normal Euler number. Proposition 1 is easily deduced from this and so is the following lemma, closely related to surgery of two-fold branched coverings considered by O. Viro [V] and J. M. Montesinos [Mo].

**Lemma (on real logarithmic transformation).** Let $E \to B$ be an elliptic fibration commuting with antiholomorphic involutions $c: E \to E$ and $\sigma: B \to B$. Let $F$ be a fibre with $c(F) = F$ and $F \cap \text{fix}(c) \neq \emptyset$. Then there exists a logarithmic transform $E'$ of $E$ along $F$ of any given multiplicity which admits an antiholomorphic involution extending $c|_{E',F}$ with orbit space diffeomorphic to $E/c$.

For any involution $c$ of a $D_{p,q}$-surface $M$ with $M/c$ diffeomorphic to $S^4$, the topology of the fixed point set and its normal Euler number are determined by the topology of $M$, and thus $\text{fix}(c)$ is again $\#_{10}RP^2$ embedded in $S^4$ with normal Euler number 16. Under appropriate conditions we can also control the fundamental group of the complement of the fixed point set in $S^4$.

**Proposition 2.** For any odd $q$ there exists a $D_{p,q}$-surface $M$ and an involution $c$ as in Proposition 1 with abelian $\pi_1((M/c) - \text{fix}(c))$ (implying $\pi_1((M/c) - \text{fix}(c)) = \mathbb{Z}_2$).
We will indicate the proof of this proposition in the next section. For $p = 2$ and $q = 2n + 1$ let us take such $M$ and $c$ and denote by $F_n$ the image of $\text{fix}(c)$ under some diffeomorphism $M/c \rightarrow S^4$. Since we can get $M$ back from $F_n$ as a 2-fold covering of $S^4$ branched along $F_n$, the results of [OV, FM] imply that for $n \neq m$ pairs $(S^4, F_n)$, $(S^4, F_m)$ are not diffeomorphic. This result together with the following proposition implies our theorem.

**Proposition 3.** Let $S$ be a fixed nonorientable 2-manifold and $k$ a fixed integer. Consider pairs $(S^4, S)$, $S$ a smooth submanifold of $S^4$ with normal Euler number $k$ and $\pi_1(S^4\setminus S) = \mathbb{Z}_2$. Choose for each pair a smooth isomorphism of a tubular neighborhood of $S$ with a fixed 2-dimensional disk bundle over $S$ with normal Euler number $k$ and identify all boundaries of these tubular neighborhoods by them.

Then the number of homeomorphism types rel. boundary of the complements of the tubular neighborhoods is finite.

We hope to prove that all $(S^4, F_n)$ are homeomorphic, extending a result of T. Lawson [L], who showed that if $S = \mathbb{R}P^2$ there is a unique homeomorphism type of such knottings. At present, in our proof of Proposition 3, which uses the surgery method of [Kr] (applicable in dimension 4 by Freedman’s results [F2]), there occur several obstructions sitting in nontrivial finite groups. We don’t see an obvious reason for them to be trivial.

![Figure 1](image1)

**Figure 1**

3. Knotting constructions. Let $X$ be a smooth 4-manifold and $F$ a smooth closed 2-submanifold of $X$. Let $M \subset X$ be a membrane homeomorphic to $S^1 \times I$ with $\partial M = M \cap F$ and let $M$ have index 0 or, equivalently, there exists a diffeomorphism of a regular neighborhood $N$ of $M$ in $X$, $\varphi: N \rightarrow S^1 \times D^3$, mapping $N \cap F$ onto $S^1 \times (I \cup I)$, and such that the segments $I \cup I$ are embedded unknotted and unlinked into $D^3$ as in Figure 1. (II means the disjoint sum operation.) For arbitrary relatively prime $p, q$ denoted by $K_{p,q}(F, M, \varphi)$ a new smooth submanifold of $X$ obtained from $F$ by replacing the embedded segments $I \cup I \hookrightarrow D^3$ drawn in Figure 1 by the two embedded segments in Figure 2.

**Proposition 4.** If $F \setminus \partial M$ is connected and $\pi_1(X\setminus (F \cup M))$ is abelian, then $\pi_1(X\setminus K_{2,q}(F, M, \varphi))$ is abelian for any odd $q$. 

![Figure 2](image2)
PROPOSITION 5. Let $T$ be a nonsingular fibre of a real (i.e., equivariant with respect to the standard complex conjugations) elliptic fibration $M \to \mathbb{CP}^1$, $M$ a $D_{1,1}$-surface. Let $T$ be invariant under $c$ and intersect the fixed point set $F$ of $c$ in two disjoint circles. Then the 2-fold covering of $S^4 = M/c$ branched over $K_{p,q}(F,T/c,\varphi)$ for some $\varphi$ is equivariantly diffeomorphic to a $D_{p,q}$-surface.

In the situation of Proposition 5 the conditions of Proposition 4 are satisfied as can be shown by the method of [Fi]. Thus Propositions 4 and 5 imply Proposition 2.

REFERENCES


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