Other representations of solutions are mentioned, for instance those in terms of generalized hypergeometric functions. The G-functions have one decisive advantage over these alternatives: The differential equations have irregular singularities at infinity. There, different solutions may have different orders of magnitude. In linear combinations of these solutions the fastest-growing component present in them determines their order of magnitude. In the solutions by G-functions the several orders of magnitude appear separately, while in the solutions by generalized hypergeometric functions these distinctions are obliterated by the presence of a contribution from the most dominant solution.

The last third of the book deals with applications. Some of them originate in physics, such as boundary layers in magnetohydrodynamics, plasmas, stellar winds and viscous flows. Other applications are purely mathematical. They concern the spectral theory of differential operators in Hilbert space, in particular the extension to higher order of differentiation of the Titchmarsh-Weyl theories on the existence and number of L^2-eigenfunctions.

I believe that this book will be often useful to readers who are looking for ways to deal with some particular differential equation of order higher than two. Rarely will it be studied from beginning to end. The task of following a thread through the book to the formulas and techniques needed in the study of some specific equation would have been made easier if the displayed formulas had been printed in a more easily readable type.

It is often possible to construct transformations that reduce differential equations of a general class into equations of the special forms analyzed in this book. This "comparison" technique is well developed for second-order equations. I would be pleased if this book stimulated the search for more general transformations of this kind.

WOLFGANG WASOW

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Manifolds of nonpositive curvature, by Werner Ballmann, Mikhael Gromov and Viktor Schroeder. Progress in Mathematics, vol. 61, Birkhäuser, Boston, Basel, Stuttgart, 1985, iv + 263 pp., $37.00. ISBN 0-8176-3181-X

This book grew out of four lectures delivered by Mikhael Gromov in 1981 at the College de France in Paris. Its purpose is twofold, namely to give an introduction to manifolds of nonpositive curvature and to give the proof of two outstanding results: the rigidity of locally symmetric spaces in the class of all manifolds of nonpositive curvature (in generalization of Mostow's rigidity theorem), as well as an estimate for the topology of nonpositively curved *analytic* manifolds of finite volume (for more precise statements see below).

Viktor Schroeder has worked out and written up these lectures and enriched them by several appendices which contain complementary material as well as his own contributions. Finally there is an appendix by Werner Ballmann on recent developments in the theory by which in particular Gromov's rigidity theorem can be deduced from that of Mostow.

The most important invariant of a Riemannian manifold is its curvature, more precisely its sectional curvature, which is measured at each point for each two-dimensional subspace of the tangent space at that point. As usual this will be denoted by K. The sign of the curvature has a strong influence on the topology and geometry of the manifold. For a surface in space, K coincides with the Gaussian curvature, i.e., with the product of the two principal curvatures, and is positive at points where the surface is convex and negative where it looks like a saddle. This implies in particular that there are no compact surfaces with $K \leqslant 0$ in $\mathbf{R}^3$. In fact, at a point which is farthest away from the origin the surface looks necessarily convex and thus has strictly positive curvature there.

Nevertheless there are plenty of examples of manifolds with strictly negative curvature ($K < 0$) and even more with nonpositive curvature ($K \leqslant 0$). The uniformization theorem shows for instance that any compact orientable surface of genus $g \geqslant 2$ admits a metric of constant curvature -1 and in fact a whole variety: the isometry classes of such metrics are in 1-1 correspondence to the conformal structures on that surface and thus form the Teichmüller space T^g which has dimension $6g - 6$. As another example, the complement of most knots in $\mathbf{R}^3 \cup \{\infty\} = S^3$, e.g., the figure-eight knot admits a hyperbolic structure, i.e., a complete metric of finite volume and constant curvature -1, as has been shown recently by W. Thurston.

Manifolds of constant negative curvature belong to the broad class of locally symmetric spaces (of noncompact type) whose universal coverings are the symmetric spaces. These are among the most beautiful examples of Riemannian manifolds because they possess a lot of symmetries. For each point the reflection at the point along the geodesics is a globally defined isometry. One of their advantages is that they can be described algebraically as a homogeneous space G/K. In the noncompact case G is a semisimple Lie group without compact factor and center and K a maximal compact subgroup. The hyperbolic spaces over the reals, the complex numbers, the quaternions, and the Cayley hyperbolic plane are precisely those of strictly negative curvature. These form only the tip of an iceberg, all others (of noncompact type) have curvature $K \leqslant 0$ but possess also zero curvatures. In fact, they are of rank $\geqslant 2$ which means that each geodesic is contained in a flat totally geodesic subspace of dimension at least two. It is this huge number of flat subspaces which makes a (locally) symmetric space of rank $\geqslant 2$ so rigid.[1]

As has been shown by Borel, using methods from number theory, every symmetric space of noncompact type admits infinitely many compact quotients as well as noncompact ones of finite volume. Thus there are plenty of

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[1] This is not quite correct since in the dual case of nonnegative curvature there are plenty of examples of nonrigid higher rank manifolds. These can be constructed by taking a normal homogeneous space G/H with $\dim H \leqslant \operatorname{rank} G - 2$.

examples of manifolds of nonpositive curvature and one gets even more by applying certain constructions to them like gluing them together along appropriate ends or taking ramified coverings. For the latter construction see the recent paper of Gromov and Thurston **[Gr-Th]**.

Intrinsically, the sign of the curvature may be expressed by the velocity by which two geodesics move apart. Roughly speaking, the more negative the curvature becomes the faster they diverge. By comparison with $\mathbf{R}^n$ it follows that geodesics emanating from a point in a simply connected manifold of nonpositive curvature never come together again. This is the essential reason for the classical theorem of Hadamard and Cartan:

THEOREM. *Let M be a complete manifold of nonpositive curvature and $p \in M$. Then the exponential map* $\exp_p\colon T_pM \to M$ *(which maps the rays through* 0 *onto geodesics through p with the same initial direction) is a covering map.*

Thus, the universal covering of M may be identified with T_pM and is in particular contractible. Therefore the higher homotopy groups of M vanish: $\pi_k(M) = 0$ for $k \geqslant 2$ and the fundamental group $\pi_1(M)$ is the most important topological invariant and plays a decisive role. It probably determines the topology and geometry of M to a great extent, although it is not quite clear up to now how much this is the case. However, there are many positive results in this direction and the most famous one is the rigidity theorem of Mostow which, in a slightly simplified form, can be stated as follows.

THEOREM (MOSTOW). *Let M_1 and M_2 be two compact locally symmetric spaces of equal volume with irreducible universal coverings and of dimension bigger than two. Then M_1 and M_2 are isometric if and only if they have isomorphic fundamental groups.*

The two-dimensional case has to be excluded since there is the Teichmüller space of different metrics of constant curvature −1. One of the main goals of the book is to prove the following generalization.

THEOREM. *Let M_1 and M_2 be two compact manifolds of nonpositive curvature of equal volume and with irreducible universal coverings. Assume that M_1 is locally symmetric of rank* $\geqslant 2$. *Then M_1 and M_2 are isometric if and only if they have isomorphic fundamental groups.*

A crucial role in the proof of both theorems is played by the Tits building of a symmetric space. Geometrically speaking it describes the intersection pattern of the flats, i.e., of the flat totally geodesic subspaces of maximal dimension. Since this is rather complicated in the manifold itself, one considers it at infinity, i.e., in the boundary, which can be attached to any complete simply connected manifold of nonpositive curvature in a natural way. By a fundamental result of Tits, the building determines a symmetric space of rank $\geqslant 2$ essentially up to isometry. Therefore, the main idea in Mostow's proof is to construct from an isomorphism between the fundamental groups an isomorphism between the Tits buildings, via a lift of a homotopy equivalence to the universal coverings $\tilde{M}_i$. Now, in Gromov's situation only $\tilde{M}_1$ is a priori symmetric and has a nice Tits building. Gromov overcomes this difficulty by attaching to each point of $\tilde{M}_2$ an involution of the Tits building of the symmetric space $\tilde{M}_1$. The associated isometry has a unique fixed point and thus gives a map between $\tilde{M}_2$ and $\tilde{M}_1$. This turns out to be an isometry which finally can be pushed down to the quotients. Shortly after Gromov's lectures at Paris, W. Ballmann, M. Brin, K. Burns, P. Eberlein, and R. Spatzier obtained other deep results about manifolds of nonpositive curvature in a series of papers. Finally, Ballmann proved **[B]**:

THEOREM. *Let M be a complete manifold of nonpositive curvature of finite volume and rank* $\geqslant 2$ *and with irreducible universal covering. Then M is locally symmetric.*

It follows easily from this theorem that in Gromov's situation both manifolds are locally symmetric, so that Mostow's rigidity theorem can be applied.

The second main result of the book concerns the topology of analytic manifolds of nonpositive curvature. It extends known results for locally symmetric spaces as well as earlier results of Gromov himself about manifolds of strictly negative curvature.

THEOREM. *Let M be a real analytic complete manifold of finite volume and of bounded nonpositive curvature,* $-k \leqslant K \leqslant 0$.

(i) *M is of finite topological type, i.e., diffeomorphic to the interior of a compact manifold with boundary. In particular it has only finitely many ends.*

(ii) *If the universal covering of M does not have a euclidean factor then*

$$\sum_i b_i \leqslant c_n k^{n/2} \operatorname{vol}(M),$$

where the b_i are the Betti numbers of M with respect to any coefficient field and c_n depends only on $n = \dim M$. *In particular, the volume of M is bounded from below.*

Examples show that the assumption on the analyticity is really necessary.

An essential ingredient for the complicated typical Gromov-style proof is the so-called Margulis lemma, which had been used by Gromov already several times very successfully.

THEOREM (MARGULIS). *For every n there exists an* $\varepsilon_n > 0$ *such that for any complete, simply connected n-dimensional manifold with curvature K,* $-1 \leqslant K \leqslant 0$, *and any discrete group Γ of isometries the subgroup $\Gamma_\varepsilon(x)$ generated by* $\{\gamma \in \Gamma \mid d(\gamma\chi, x) < \varepsilon_n\}$ *is almost nilpotent, i.e., has a nilpotent subgroup of finite index.*

A version of this lemma for manifolds of strictly negative curvature together with finiteness results for the topology and an upper bound for the volume is also contained in **[H]**.

Roughly speaking, the Margulis lemma enables one to get information about the topology of a metric ball if the injectivity radius becomes very small. In the case of strictly negative curvature, for example, it shows that the manifold can be covered by a certain number of convex balls or sets homeomorphic to vector bundles over S^1, which then yields by a refined Mayer-Vietoris argument the estimate for the Betti numbers.

The book contains much more than the mere proof of the two principal results. In the first two chapters Gromov explains basic facts about manifolds of nonpositive curvature and of course does this in his very own original style. Therefore this introduction contains exciting new ideas even for experts. Certainly the most important one is the definition of the Tits metric for points at infinity of a complete simply connected manifold of nonpositive curvature. If x and y are such points then $\sphericalangle(x, y)$ is defined as the supremum of all angles under which x and y can be seen from a finite point. The Tits distance Td is then defined as the corresponding inner metric, i.e. as the infimum of the length (w.r.t. $\sphericalangle$) of all curves in the boundary connecting the points. One of its main features is that flat totally geodesic subspaces are reflected by this metric. They give rise to isometrically embedded round spheres in the boundary. In particular, the boundary of $\mathbf{R}^n$ with the Tits metric is the standard sphere S^{n-1}. For a symmetric space, Tits metric and Tits building contain the same information and determine each other. This explains the name. There is no doubt that the Tits metric will be an important tool for further investigations.

The book by M. Gromov, V. Schroeder, and W. Ballmann is an extraordinary one which contains a wealth of new ideas as well as plenty of inspiration for further work. (For example the study of the relation between Tits metric and fundamental groups of compact quotients seems to be very promising.) It is of course not a textbook in the usual sense. Although it starts quite easy it ends more or less like a research paper. But due to the excellent work of V. Schroeder the presentation is always clear. I think the book will have a strong influence on the further development of the theory of manifolds of nonpositive curvature.

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ERNST HEINTZE

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Diophantine inequalities, by R. C. Baker, London Mathematical Society Monographs, New Series, vol. 1, Clarendon Press, Oxford, 1986, xii + 275 pp., $65.00. ISBN 0-19-853545-7

The volume under review deals with analytic methods for diophantine inequalities. For diophantine inequalities, and in fact for many diophantine problems in general, the appropriate tool is Fourier analysis. Among the most