2 F. H. Clarke and R. B. Vinter, *Existence and regularity in the small in the calculus of variations*, J. Differential Equations **59** (1985), 336–354.

3. F. H. Clarke and P. D. Löwen, *An intermediate existence theory in the calculus of variations* (to appear).

4. V. M. Zeidan, *Sufficient conditions for the generalized problem of Bolza*, Trans. Amer. Math. Soc. **275** (1983), 561–586.

JOHN L. TROUTMAN

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 2, April 1988

0273-0979/88 $1.00 + $.25 per page

Quadratic forms and Hecke operators, by Anatolij N. Andrianov. Grundlehren der mathematischen Wissenschaften, vol. 286, Springer-Verlag, Berlin, Heidelberg, New York, 1987, xii + 374 pp., $102.00. ISBN 3-540-15294-6

This book is an important contribution to analytic number theory, especially to the branch which is collected in *Mathematical Reviews* under the title "Modular Theory". Before we come to our subject we have to tell something of the background in which Hecke's theory assumes a conspicuous place. In this we allow ourselves some simplifications.

A question which has attracted mathematicians since Fermat and Euler is the following: given a positive definite quadratic form

$$f[x] = \sum_{i,j=1}^{2k} f_{ij} x_i x_j$$

with integral f_{ij}, how many solutions has the equation

$$f[x] = g \tag{1}$$

for an integral g and integral x_i? The question can be put into the frame of analytic function theory by the evident identity

$$\vartheta(z,f) = \sum_{x \in \mathbf{Z}^{2k}} e^{2\pi i z f[x]} = \sum_g N(f,g) e^{2\pi i g z} \tag{2}$$

with a complex variable z, where $N(f,g)$ denotes the number of solutions of (1). What makes the problem interesting is the fact that this function is a modular form in z satisfying the functional equations

$$\varphi\left(\frac{az+b}{cz+d}\right)(cz+d)^{-k} = \pm\varphi(z) \tag{3}$$

where $a, \ldots, d$ are integers with the properties

$$ad - bc = 1, \qquad c \equiv 0 \mod q \tag{4}$$

and where q is an integer, characteristic for $f[x]$, the so-called *level* of f. The matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying (4) form the subgroup $\Gamma_0(q)$ of the modular group $\mathrm{SL}_2(\mathbf{Z})$.

Modular forms and their quotients, modular functions, are intrinsically connected with elliptic functions and so belong to the central subjects of classical mathematics until our day. They have contributed fruitful ideas to general and especially algebraic number theory.

In the middle thirties Hecke considered the linear spaces $\mathcal{M}(k, g)$ of modular forms satisfying the functional equations (3). These spaces have finite dimensions. Hecke introduced the following linear operators mapping $\mathcal{M}(k, q)$ into itself:

$$\varphi(z)|T_n = \pm n^{k-1} \sum_i \varphi\left(\frac{a_i z + b_i}{c_i z + d_i}\right)(c_i z + d_i)^{-k},$$

where $\det\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) = n$, $c_i \equiv 0 \bmod q$ and $(n, q) = 1$, and where the matrices $M_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ represent all double cosets $\Gamma_0(q) M_i \Gamma_0(q)$ with determinant n. These *Hecke operators* T_n generate a commutative semisimple ring over $\mathbf{Z}$, the *Hecke ring*, and satisfy the equations (where n_1, n_2 are prime to the level q)

$$T_{n_1} T_{n_2} = \sum_{d|(n_1, n_2)} d^{k-1} T_{n_1 n_2 d^{-2}}, \tag{5}$$

particularly for $(n_1, n_2) = 1$: $T_{n_1} T_{n_2} = T_{n_1 n_2}$.

The spaces $\mathcal{M}(k, q)$ are spanned by the eigenfunctions

$$\varphi_i(z)|T_n = \lambda_i(n)\varphi_i(z) \quad \text{for } (n, q) = 1. \tag{6}$$

They have Fourier expansions

$$\varphi_i(z) = \sum_n \lambda_i(n)^{2\pi i n z} \tag{7}$$

whose coefficients are eigenvalues of T_n for $(n, q) = 1$.

The multiplicative properties (5) allow a surprising expression. Attach to the Fourier series (7) the Dirichlet series

$$\varsigma_i(s) = \sum_n \lambda_i(n) n^{-s} \tag{8}$$

(which actually can be obtained from (7) by the Laplace operator). Then the equations (5) imply the Euler product

$$\varsigma_i(s) = \prod_p \varsigma_{ip}(s) \tag{9}$$

extended over all primes p whose factors are rational functions in p^{-s}, and especially

$$\varsigma_{ip}(s) = (1 + \lambda_i(p) p^{-s} + p^{k-1-2s})^{-1} \tag{10}$$

for all p which are prime to the level q.

Particular eigenfunctions of the Hecke operators are the following: let $f_i[x]$ represent all classes of quadratic forms of a genus and denote their number of (integral) isometries with e_i. Then

$$\varphi(z) = \sum_i e_i^{-1} \vartheta(z, f_i) \tag{11}$$

is expressible as the so-called *Eisenstein* series

$$E(z,\chi) = \sum \frac{\chi(d)}{(cz+d)^k} \tag{11a}$$

summed over all $c, d \in \mathbf{Z}$ with $c \equiv 0 \pmod q$. They are eigenfunctions of all Hecke operators T_n with $(n,q) = 1$, and their Fourier coefficients $\lambda_i(n)$ are known sums of divisors of n multiplied with certain roots of unity. We mention only one particular case: the form

$$f[x] = x_1^2 + \cdots + x_4^2$$

has level 4, and it is the only representation of its genus. In this case Hecke's theorem states: The number $N(f,g)$ of representations of an odd $g = p_1^{n_1} p_2^{n_2} \cdots$ is $N(f,g) = 8p_1^{n_1-1}(p_1+1)p_2^{n_2-1}(p_2+1)\cdots$. This fact, discovered by Jacobi 140 years ago, is one of the historic roots of Hecke's theory.

It is important to mention that all modular forms of weights $k > 1$ are linear combinations of either theta series (2) or certain derivations of such series (Hijikata, Pizer, Shemanske). Therefore Hecke's theory concerns chiefly quadratic forms. This fact can be accentuated even more by the following. Let $f_i[x]$ represent all classes of quadratic forms of a given genus and consider the similarity transformations

$$N_{ij}^t(f_i)N_{ij} = n(f_j), \tag{12}$$

where (f_i) means the matrix of the form $f_i[x]$. Furthermore let

$$\vartheta(z, f_i)|T_n = \sum_j t_{ij}(n)\vartheta(z, f_j)$$

express the action of the Hecke operator T_n on the theta functions. Then, in many cases, $t_{ij}(n)$ is equal to the number of matrices N_{ij} in (12) for given n and ij.

We observe here, speaking with Leibniz, a "prestabilized harmony" between two domains: function theory and number theory.

About the same time as Hecke's theory there appeared Siegel's papers on the analytic number theory of quadratic forms. Generalizing (1), Siegel; considered the equations

$$X^t(f)X = (g) \tag{13}$$

where the square matrices (f) and (g) have $2k$, resp. h, rows and X are matrices of $2k$ columns and h rows. (1) is the special case of (13) with $h = 1$. He introduced medium p-adic measures $N_p(f,g)$ of solutions of (13), including a real medium measure $N_\infty(f,g)$. Now let $f_1, f_2, \ldots$ represent all classes of a given genus and $E(f_i)$ be the integral solutions of (13) with $g = f_i$. Then Siegel's main theorem states:

$$\frac{N(f_1,g)E(f_1)^{-1} + N(f_2,g)E(f_2)^{-1} + \cdots}{E(f_1)^{-1} + E(f_2)^{-1} + \cdots} = N_\infty(f,g)\prod_p N_p(f,g). \tag{14}$$

The $N_p(f,g)$, including the case $p = \infty$, are equal for all f_i.

This theorem allows an analytic formulation: The theta series and its Fourier expansion

$$\vartheta(Z,f) = \sum_X e^{2\pi i\sigma(ZX^t(f)X)} = \sum_g N(f,g)e^{2\pi i\sigma(Zg)} \tag{15}$$

is a *Siegel modular form*, satisfying

$$\vartheta((AZ+B)(CZ+D)^{-1}, f)\det(CZ+D)^{-k} = \chi\begin{pmatrix} A & B \\ C & D \end{pmatrix}\vartheta(Z,f). \tag{16}$$

In (15) σ means the trace, and

$$Z = X + iY$$

is a symmetric complex matrix of h rows whose imaginary part is the matrix of a semidefinite quadratic form. In (16)

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a matrix of $2k$ rows with integral elements with the properties

$$M^tJM = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} J \begin{pmatrix} A & B \\ C & D \end{pmatrix} = J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \tag{17}$$

and

$$C \equiv 0 \mod q \tag{17a}$$

(the *level* of f). The factor $\chi\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a root of unity. The matrices M with these properties form the group $\Gamma_0(q)^h$, the *Siegel modular congruence group of degree h and level q*. The sum

$$\frac{\vartheta(Z,f_1)E(f_1)^{-1} + \vartheta(Z,f_2)E(f_2)^{-1} + \cdots}{E(f_1)^{-1} + E(f_2)^{-1} + \cdots} = E(Z) \tag{18}$$

is the generalization of the Eisenstein series (11).

Siegel once remarked that he left to the younger generation the discovery of a large deposit of mathematical truths. It was clear from the beginning, that one should find an analogue of Hecke's operators. Indeed, Sugawara soon gave such analogues in two brief notes. But the further development was held back by the war. Since 1950 the work on Siegel's mine was taken up vigorously. Among the foremost workers was Maass. The famous Cartan Seminar 1957/58 played an important role. Further steps were taken by Satake, Freitag, and others, especially Andrianov and his Leningrad school. Unfortunately much of the material brought to the surface was unwieldy and needs to be processed.

But, as the author states in this book: "The theory has reached a certain maturity, and the time has come to give an up-to-date report in concise form, in order to provide a solid ground for further development."

Indeed, two such reports are given, by E. Freitag [**3**] and the present book by Andrianov, which we will now discuss in more detail, but with a few simplifications. While Freitag is chiefly interested in the algebraic nature of the field of modular functions and the rings of modular forms, Andrianov aims at an extension of Hecke's theory to Siegel modular forms. The two books do not intersect much.

Andrianov's book introduces a reader with only basic knowledge of function theory and number theory to his and his pupils' work. His first step is therefore to prove that Siegel theta series (15) satisfy the functional equations (16). The proof follows and generalizes a procedure proposed by the reviewer [**2**] which was also observed by Freitag [**3**]. It consists of two steps: in the first the factor $\chi\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is left open, in the second it is shown to consist principally of a certain Gaussian sum. The reviewer thinks that it can be simplified and abbreviated even more.

Chapter Two contains the necessary basic material of the theory of Siegel modular groups and modular forms. We mention particularly the fundamental domain, the Petersson scalar product, and the Siegel Φ-operator which maps the spaces of modular forms of degree n onto those of smaller degree, in short

$$\mathcal{M}^n(k,q)|\Phi = \mathcal{M}^{n-1}(k,q).$$

The third chapter is devoted to an extensive study of the Hecke rings for their own sake. This had been done for the first time by Shimura in the case of the full Siegel modular group $\Gamma^n = \mathrm{Sp}_n(\mathbf{Z})$ and is now common use. The Hecke ring is generated over $\mathbf{Z}$ by the sums on the double cosets $\sum_i \Gamma^n \gamma_i \Gamma^n$ under the assumption that $\gamma_i^{-1}\Gamma^n\gamma_i \cap \Gamma^n$ has finite index in Γ^n. The aim is to find prime elements and the decomposition of more general elements as products of prime elements. Hecke rings have infinite dimension over $\mathbf{Z}$, and their properties differ from those of more familiar rings of algebraic numbers, for instance. The author restricts himself to groups Γ^n which are congruence subgroups of $\mathrm{Sp}_n(\mathbf{Z})$, particularly $\Gamma^n = \Gamma_0^n(q)$ and $\Gamma^n(q)$ (for which both $B \equiv C \equiv 0 \bmod q$).

In Chapter Four the *Hecke operators* are applied to modular forms

$$\begin{aligned} F(Z)|^k T &= \sum_i F(Z)\Big|^k \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \\ &= \sum_i F((A_iZ+B_i)(C_iZ+D_i)^{-1})|C_iZ+D_i|^{-k} \end{aligned}$$

which map the spaces $\mathcal{M}^n(k,q)$ into themselves, under certain conditions. Under these the abstract Hecke rings are mapped on the rings of actions of the Hecke operators which now have finite dimensions over $\mathbf{Z}$. One is inclined to conjecture that the knowledge of the traces of the Hecke operators would reveal much of their nature. But these traces are known only for $n = 1$ and 2, and for $n = 2$ they are immensely complicated and perhaps of no use.

We mention four important results of the fourth chapter.

1. The Žarkowskaya commutation relation: there is a surjective homomorphism $\Psi^n \to \Psi^{n-1}$ of the Hecke rings of degree n and $n-1$ such that

$$F_k^n|\Psi^n|\Phi = F_k^n|\Phi|\Psi^{n-1} \subseteq F_k^{n-1}|\Psi^{n-1}$$

2. The Hecke operators are symmetric with respect to the scalar product. Consequently a space of modular forms stable under all Hecke operators has a basis consisting of eigenforms.

3. Now the way is open to approach Hecke's theory and its extension to $n > 1$. The classical case $n = 1$ is briefly presented with proofs. The chief result for $n = 2$ can be described as follows: Let

$$F(Z) = \sum_A N(A)e^{\pi i \sigma(AZ)} \tag{19}$$

be the Fourier expansion of an eigenfunction of all even Hecke operators (even = certain subrings of Hecke rings) in the space $\mathcal{M}^n(k,q)$ of modular forms with respect to $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $B \equiv C \equiv 0 \bmod q$. To (19) attach the Dirichlet series

$$D(s, \Delta) = \sum_{A_i} \eta(A_i) \sum_{a,m} N(maA_i)m^{-s} \tag{20}$$

where A_i runs only over the matrices representing the norm forms of ideals with respect to orders $\mathcal{O}$ in a fixed imaginary quadratic field $\mathbf{Q}(\sqrt{\Delta})$, $\Delta < 0$, and $\eta(A_i)$ means a character of the class group of $\mathcal{O}$. This Dirichlet series has an Euler product whose factors are given rational functions of p^{-s}. The latter are expressions of the local behavior of the Hecke ring. (The author proves a slight generalization of this.) This is a particularly beautiful result of the author and a high point in the book.

4. Also for $n > 2$ Dirichlet series can be attached to eigenfunction (19). Again they are formed only with the subsums of (19) for which the matrices A are similar over $\mathbf{Q}$:

$$M^t A_1 M = \alpha A_2$$

with $M \in M^n(\mathbf{Q})$. The Dirichlet series also have Euler products.

The results of Chapter Four are applied in Chapter Five to spaces of modular forms generated by theta series of quadratic forms provided that Hecke operators map theta series onto linear combinations of theta series. The proof that this is the case under certain assumptions is long and tedious. It is at first carried out for series $\vartheta(Z, f)$, where Z and (f) have the same number $n = m$ of rows. The case $n > m$ has to follow. The result is too involved to be presented here. It should be mentioned here that Freitag limits himself to theta series (15) with unimodular matrices (f), where the proof is much easier.

The coefficients of linear combinations of theta series occurring in this way are similar to those introduced by the reviewer in his 1952 book on quadratic forms [1]. In this chapter the reviewer would have liked to see special and easy examples treated, e.g. for $n = 1$ and the group $\Gamma_0(q)$.

Having come so far we may ask: what has been achieved and what is left to be done? Clearly the author has attained his aim to prove multiplicative properties of Fourier coefficients which comprise Hecke's as special cases. At first sight it looks disappointing that the final theorem does not give properties of a Dirichlet series formed with the full Fourier series (19). But it may be that such a statement cannot be expected. The advantage of the case $n = 1$ is that all one-row matrices belong to the same similarity class.

Another disappointment is the extremely complex and complicated nature of Chapters III–V. The reviewer did not expect an easier presentation; that would not have been possible today. But the subject should be meditated upon over and over again, and the author has made the first step.

Two closing remarks: A reader who only wants to understand the chief results can use the index at the end. The reviewer would like it to be even more extensive.

The book is full of interesting exercises. These are in fact often more than that, as they contain statements which are essential parts of the theory. See for instance Exercise 2.5.6. In Exercises 4.3.4–4.3.12 the reader is even guided to develop parts of Hecke's theory.

There are three appendices: (1) Elementary properties of symmetric matrices over fields. (2) The geometry of metric spaces as another expression of the theory of quadratic forms. (3) Modules and ideals in quadratic fields $\mathbf{Q}(\sqrt{\Delta})$ and their norm forms. This is helpful for the understanding of the Euler products occurring in Chapter Four.

REFERENCES

1. M. Eichler, *Quadratische Formen und orthogonale Gruppen*, 2nd ed., Springer-Verlag, Berlin, Heidelberg, New York, 1974.

2. ——, *Einführung in die Theorie der Algebraischen Zahlen und Funktionen*, Birkhäuser-Verlag, Basel and Stuttgart, 1963=*Introduction to the theory of algebraic numbers and functions*, Academic Press, New York and London, 1966.

3. E. Freitag, *Siegelsche Modulfunktionen*, Springer-Verlag, Berlin, Heidelberg, New York, 1983.

MARTIN EICHLER

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 18, Number 2, April 1988

0273-0979/88 $1.00 + $.25 per page

Arithmetic functions and integer products, by P. D. T. A. Elliott. Grundlehren der Mathematischen Wissenschaften, vol. 272, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1985, xv + 461 pp., $64.00. ISBN 0-387-96094-5

Introduction to arithmetical functions, by Paul J. McCarthy, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1986, vi + 365 pp., $35.50. ISBN 0-387-96262-X

1. The theory of numbers: its great conjectures. Problems in number theory have fascinated generations of professional and amateur scientists. Still today mathematicians are attracted to number theory because its history has brought so many conjectures. Some, like the Riemann Hypothesis, stated in 1859 (see §2), and the Goldbach conjecture, which goes back to 1742 (see §6), have yet to be proven. Others, thanks to the ingenuity of contemporary mathematicians or to highly sophisticated computer methods, have been resolved: such is the case of the Mertens conjecture (see §5), which was proven false by Odlyzko and te Riele [**39**] in 1983, some 86 years after it was stated.

Many problems in number theory involve arithmetical functions. Our intent here is to present a survey of (what we feel are) the most significant results in the theory of arithmetical functions, thereby leading us into a review of the books of McCarthy and Elliott. Though our presentation obviously cannot be exhaustive, our objective is to display most of the classical arithmetical functions (those which "made history") and to introduce the reader to the methods used by mathematicians to analyze their behavior. The two books under review are mainly concerned with results and methods in elementary and analytic number theory, though the second assumes some knowledge of probabilistic number theory; thus our survey will reflect the development of arithmetical functions only in these three areas.