CONSTRAINED POISSON ALGEBRAS AND STRONG HOMOTOPY REPRESENTATIONS

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A Poisson algebra is a commutative associative algebra $A$ with an (anticommutative) bracket $\{ , \}$ which is a derivation with respect to the commutative product: $\{f, gh\} = \{f, g\}h + f\{g, h\}$. Constraints constitute a distinguished set of elements $\phi_\alpha$ of $A$. They are said to be first class constraints if the ideal $I$ they generate (under the commutative product) is closed under Poisson bracket; $I$ need not be an ideal with respect to $\{ , \}$. This structure arises in physics with $A = C^\infty(W)$ for some symplectic manifold $W$. The constraints determine a subvariety $V \subset W$, the zero locus of $I$, and a foliation $F$ of $V$, by the flows determined by the derivations $\{ , \}$. One wishes to compute the ad $I$-invariant functions on $V$, which would give $C^\infty(V/F)$ were the foliation to give a submersion $V \to V/F$ onto a manifold.

In a remarkable series of papers, Fradkin, Batalin and Vilkovisky [0-3, 6] and then Henneaux [10] developed a method for calculating the ad $I$-invariant functions in $C^\infty(V) = A/I$ without passing through the quotient $A/I$. The method appeared to depend on solving certain specific, complicated equations and initially was applicable only locally and when $I$ was a regular ideal.

Using the techniques of 'homological perturbation theory' [7, 8, 9], I am able to justify their machinery in terms of the algebra alone, including, with Henneaux [11], the case of nonregular ideals [0]. The idea for this approach owes a great deal to the paper of Browning and McMullan [4], which revealed the structure of a multicomplex implicit in Fradkin et al and Henneaux.

The Lie algebra cohomology $H^0(I, A/I)$ computes the ad $I$-invariant functions on $V$, but physics requires a description in terms of $A$ and prefers to use $\Phi$, the linear span of the constraints $\phi_\alpha$, rather than the full ideal $I$. An obvious step algebraically is to replace $A/I$ by a free resolution over $A$. To combine this with the restriction to $\Phi \subset I$ is more subtle.

The Lie algebra cohomology of Cartan, Chevalley and Eilenberg [5] begins with the algebra $\text{Alt}(I, A/I)$ of alternating multilinear functions on $I$ with values in $A/I$ and a differential $\text{Alt} \to \text{Alt}$ (which increases the number of variables by one) given in terms of the bracket on $I$ and the adjoint representation of $I$ on $A/I$: For example, for $h: I \to A$, we have

$$(\delta h)(f, g) = h(\{f, g\}) - \{f, h(g)\} + \{g, h(f)\}.$$ 

The subalgebra $\text{Alt}_A(I, A/I)$ of $A$-multilinear functions is in fact a subcomplex with the same $H^0$. (This is isomorphic to the complex which defines
the Rinehart cohomology of the \((A/I, R)\)-Lie algebra \(I/I^2\) with coefficients in \(A/I\) [12].) The inclusion \(\Phi \subset I\) induces \(\text{Alt}_A(I, A/I) \to \text{Alt}(\Phi, A/I)\) and a differential also denoted \(\delta\). (This map is an isomorphism if \(I\) is regular.)

Now introduce a multiplicative resolution \(\pi: K_I \to A/I\), that is, \(K_I\) is a graded commutative differential algebra (with differential \(d\)) and \(\pi\) induces an isomorphism \(\pi^*: H_0(K_I) \to A/I\) with \(H_i(K_I) = 0\) otherwise. For example, if \(I\) is a regular ideal, take \(K_I\) to be the Koszul complex; more generally, the Tate resolution will do [14]. If we replace \(A/I\) by \(K_I\) and consider \(\text{Alt}(\Phi, K_I)\), the problem is to extend \(d\) to a differential \(D\) so as to realize the same homology as that of \(\text{Alt}(\Phi, A/I)\) with respect to \(\delta\). The major source of difficulty is that the adjoint representation of \(I\) on \(A/I\) does not lift to \(K_I\); in spite of this, we have:

**Theorem 1.** There are differentials \(\delta_i\) on \(\text{Alt}(\Phi, K_I)\) which increase the form degree by \(i\) and the resolution degree by \(i - 1\) such that \(\delta_0 = d\) and \(D = \sum \delta_i\) has \(D^2 = 0\) with \(\pi: K_I \to A/I\) inducing

\[
H^0(\text{Alt}(\Phi, K_I), D) \approx H^0(\text{Alt}(I, A/I), \delta).
\]

Our proof of the theorem uses the methods of homological perturbation theory [7, 8, 9]. Let \(\text{Der}^q K_I\) denote the derivations of \(K_I\) which increase resolution degree by \(q\). The collection \(\text{Der} K_I = \{\text{Der}^q K_I\}\) is made into a differential graded Lie algebra by using the graded commutator of derivations and the induced differential: \(\partial \theta = [d, \theta]\). We cannot, in general, find a representation of \(I\) in \(\text{Der} K_I\), but we can find a “strong homotopy representation”, meaning a family \(\Theta_i \in \text{Alt}_i(I, \text{Der} K_I)\) for \(i \geq 1\) satisfying the following relations: For \(i = 1\), \(\Theta^1(f) = \{f, \cdot\}\). For \(i > 1\), and \(\hat{f}_i = (f_0, \ldots, f_i)\),

\[
(\ast) \quad [d, \Theta^i] \hat{f}_i = \sum [\Theta^i, \Theta^k]([\hat{f}_i] + \sum (-1)^{j+k} \Theta^{i-1}((f_j, f_k), \ldots, \hat{j}, \ldots, \hat{k}, \ldots))
\]

For \(i = 1\), this is to be interpreted as \([d, \Theta^1] = 0\). Here \([\cdot, \cdot]\) is the usual induced bracket on \(\text{Alt}(V, L)\) for a vector space \(V\) and Lie algebra \(L\). The maps \(\Theta^i\) are constructed inductively, using a contracting homotopy \(s\) for \(K_I\), that is: \(sd + ds = 1 - \pi \) where \(\pi: K_I \to A/I \to A \hookrightarrow K_I\) and the map \(1 - \pi\) is the identity on \(I\). We begin by defining \(\Theta^1: I \to \text{Der}^0 K_I\) as an extension of the adjoint action of \(I\) on \(A\) as follows: By induction on the resolution degree of a generator \(x\) of \(K_I\) over \(A\), define \(\Theta^1(f)(x) = s\Theta^1(f)(dx)\). Verify directly that \((\ast)\) is valid in the form \([d, \Theta^1] = 0\). Now assume we have constructed \(\Theta^i\) for \(i < n\) to satisfy \((\ast)\). Let \(\text{RHS}\) denote the right-hand side of the equation \((\ast)\) for \(i = n\). Verify that \([d, \text{RHS}] = 0\) using \((\ast)\) and the Jacobi identity. Now define the derivation \(\Theta^n(\hat{f}_n)\) as \(s\text{RHS}\). We verify that

\[
[d, s\text{RHS}] = ds\text{RHS} + s\text{RHS}d = (ds + sd)\text{RHS} \quad \text{by induction}
\]

\[
= (1 - \pi)\text{RHS} = \text{RHS},
\]

since \(\text{RHS}\) raises resolution degree by at least \(j - 1 + k - 1\), which is into the kernel of \(\pi\) unless \(j = k = 1\). For \(n = 2\), we also use the fact that \(\Theta^1\) is an extension of the adjoint action of \(I\) on \(A\) in terms of the original Poisson bracket.
Since $\Phi \subset I$ need not be closed under the bracket, we cannot just restrict $D$ to $\text{Alt}(\Phi, K_I)$. Instead, the FBV construction in the regular case makes further use of the Poisson algebra. Notice that the Koszul resolution can be written as $A \otimes \wedge s\Phi$ where $s\Phi$ is isomorphic to $\Phi$ as a vector space, while $\text{Alt}(\Phi, K_I)$ contains the vector space dual $\Phi^* = \text{Hom}(\Phi, R)$. Extend the Poisson bracket of $A$ to all of $\text{Alt}(\Phi, K_I)$ by first defining $\{\Phi^*, s\Phi\}$ to be isomorphic to the usual dual pairing and then extending to a graded Poisson bracket by using the derivation property: $\{\omega, \eta \wedge \zeta\} = \{\omega, \eta\} \wedge \zeta + (-1)^{|\omega||\eta|} \eta \wedge \{\omega, \zeta\}$.

**Theorem 2.** There is an element $Q \in \prod \text{Alt}^p(\Phi, K_I)$ such that $D$ in Theorem 1 is given by $D = \{Q, \}$. We write $Q = \sum Q_p$ where $Q_p \in \text{Alt}^{p+1}(\Phi, K_I)$ takes values in $A \otimes \wedge^p s\Phi$. Although $D = \{Q, \}$, we do not have $\delta_i = \{Q_i, \}$ but rather $\delta_i$ is of bidegree $(i, i - 1)$, while $\{Q_i, \}$ has components of bidegree $(i, i - 1)$ and $(i + 1, i)$. To start, let $Q_0$ be the inclusion $\iota: \Phi \hookrightarrow A \hookrightarrow K_I$ so that $\{Q_0, \}|K_I$ is the Koszul differential. (This is easier to see in terms of a basis $\{\phi_a\}$ for $\Phi$, dual basis $\{\eta^\alpha\}$ for $\Phi^*$, and basis $\{\mathcal{P}_a\}$ for $s\Phi$ so that $Q_0 = \phi_a \eta^\alpha$.) Filter $\text{Alt}(\Phi, K_I)$ by $F^p = \sum_{i<p} \text{Alt}^i$, and for any element $R$ of the complex, let $R^2$ denote $\frac{1}{2}\{R, R\}$. Now construct $Q_i$ by induction so that the partial sums $R_i = \sum Q_j$ have the following properties:

$$R_p^2 \in F^{p+2} \quad \text{and} \quad dR_p^2 \in F^{p+3}.$$ Define $Q_{n+1} = -sR_n^2$. A slightly complicated computation then shows that $R_{n+1}$ satisfies the inductive hypothesis.

We have left to show that $D$ gives the desired homology. The resolution $\pi: K_I \to A/I$ induces a map of complexes. If we filter $\text{Alt}(\Phi, K_I)$ as above, the associated graded has differential just $d$ with homology $\text{Alt}(\Phi, A/I)$. A standard spectral sequence argument then gives the desired result.

Because of the motivating physics, Fradkin et al consider also the situation in which $A$ is a super-Poisson algebra, i.e. $\mathbb{Z}/2$-graded with appropriate signs throughout. Now we need to use a super-resolution, for example, Jozefiak's [13]. The formalism we have used need only be made super (i.e. attend carefully to signs) with some extra care interpreting formal power series.

As a guide to the physics literature, in the regular case, $Q_i$ corresponds to an expression $U^a_{\bar{\alpha}} \eta^b_{\bar{\beta}} \mathcal{P}_a$ where $\bar{\alpha} = \alpha_1 \cdots \alpha_{i-1}, \bar{\beta} = \beta_1 \cdots \beta_{i+1}$ and $\eta^b = \eta^{\beta_1} \wedge \cdots \wedge \eta^{\beta_{i+1}}$, etc. Finally, the $\eta^b$ are called ghosts, the $\mathcal{P}_a$ anti-ghosts and, in the nonregular case, syzygies are called extraghosts or ghosts-of-ghosts-of-

**References**


