BI-ININVARIANT SCHWARTZ MULTIPLIERS AND LOCAL SOLVABILITY ON NILPOTENT LIE GROUPS

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Let $X$ denote a finite-dimensional vector space with a fixed positive definite inner product, and let $\mathcal{S}(X)$ denote the Schwartz space on $X$. We let $\mathfrak{M}(X)$ denote the space of continuous endomorphisms of $\mathcal{S}(X)$ that commute with the action of $X$ on $\mathcal{S}(X)$. The elements of $\mathfrak{M}(X)$ are given by convolution by tempered distributions; i.e., for $E \in \mathfrak{M}(X)$ there is a $D_E \in \mathcal{S}^*(X)$ such that $Ef(x) = (D_E, l_x f) := D_E * f(x)$, where $l_x f(y) = f(y - x)$. Conversely, if $D \in \mathcal{S}^*(X)$, then one can easily see that $E_D: f \mapsto D * f$ is a mapping of $\mathcal{S}(X)$ into the smooth functions on $X$ that commutes with translation. Schwartz [S] shows that $ED \in \mathfrak{M}(X)$ if and only if $D$, the Fourier transform of $D$, is given by a smooth function on $X^*$ which has polynomial bounds on all derivatives. In this note we announce analogues of these results for arbitrary nilpotent Lie groups. Complete proofs will appear elsewhere.

Let $N$ denote a connected, simply connected nilpotent Lie group, with Lie algebra $n$. The exponential mapping, exp: $n \to N$, is a diffeomorphism, and in terms of the corresponding coordinates left and right translation on $N$ are polynomial mappings. Thus, if $\mathcal{S}(N)$ denotes the image under composition with exp of $\mathcal{S}(n)$, the right and left action of $N$ on $\mathcal{S}(N)$ are continuous endomorphisms, where $\mathcal{S}(N)$ is topologized so that composition with exp is an isomorphism from $\mathcal{S}(n)$ to $\mathcal{S}(N)$. We denote by $\mathcal{M}^*(N)$ the dual of $\mathcal{M}(N)$, the space of tempered distributions on $N$.

For $f \in \mathcal{S}(N)$, the Fourier transform of $f$, $\hat{f}$, is defined on $n^*$, the dual of $n$, by

$$\hat{f}(\xi) = \int_n f(x) e^{-2\pi i \langle \xi, x \rangle} \, dX.$$ 

One has that $f \mapsto \hat{f}$ is an isomorphism from $\mathcal{S}(N)$ onto $\mathcal{S}(n^*)$. For $D \in \mathcal{S}^*(N)$, $\hat{D}$ is defined on $\mathcal{S}(n^*)$ by $\langle \hat{D}, f \rangle = \langle D, \hat{f} \circ \log \rangle$, where log denotes the inverse of exp.

Let Ad* denote the coadjoint representation of $N$ on $n^*$. A tempered distribution $D$ on $n^*$ is said to be Ad*-invariant if $\langle D, f \circ \text{Ad}^* x \rangle = \langle D, f \rangle$ for all $x \in N$ and $f \in \mathcal{S}(n^*)$. A tempered distribution $D$ on $N$ is said to be bi-invariant if $\langle D, \tau_x^{-1} f \rangle = \langle D, l_x f \rangle$ for all $f \in \mathcal{S}(N)$, where $\tau_x f(y) = f(yx)$ and $l_x f(y) = f(x^{-1}y)$ for all $x, y \in N$. A straightforward computation shows that an element $D \in \mathcal{S}(N)$ is bi-invariant if and only if $\hat{D}$ is Ad*-invariant.

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Let $\mathcal{M}(N)$ denote the space of continuous endomorphisms on $\mathcal{P}(N)$ that commute with both right and left translations by elements of $N$. As in the Euclidean case, one has that for each $E \in \mathcal{M}(N)$ there is a $D_E \in \mathcal{P}^*(N)$ such that $Ef = D_E \ast f$, where, as before, $D_E \ast f(x) := \langle D_E, l_x \hat{f} \rangle$. If $D \in \mathcal{P}^*(N)$ we denote by $E_D$ the mapping defined on $\mathcal{P}(N)$ by $E_D f = D \ast f$.

Let $PB^\infty_N(n^*)$ denote the space of smooth, $\text{Ad}^*$-invariant functions defined on $n^*$ with polynomial bounds on all derivatives. This space is topologized using the seminorms $\nu_{ij}$ defined on $PB^\infty(n^*)$ by
\[
\nu_{ij}(\theta) = \sup_{|\alpha| \leq j} \sup_{\xi \in n^*} |\partial^\alpha \theta(\xi)|/(1 + ||\xi||^2)^i,
\]
where $\partial^\alpha$ denotes the standard differential operator corresponding to the multi-index $\alpha$, and some fixed basis of $n^*$. A sequence $\{E_n\} \subset \mathcal{M}(N)$ converges to 0 if $E_n f \to 0$ in $\mathcal{P}(N)$ for each $f \in \mathcal{P}(N)$.

**Theorem A.** The mapping $\mathcal{M}(N) \to PB^\infty_N(n^*): E \to D_E$ is a homeomorphism and an algebra isomorphism, the products being composition on $\mathcal{M}(N)$ and pointwise multiplication on $PB^\infty_N(n^*)$.

For $\xi \in n^*$, let $\pi_\xi$ denote the irreducible unitary representation of $N$ that corresponds to the $\text{Ad}^*$-orbit of $\xi$ by the Kirillov theory. For $\theta \in PB^\infty_N(n^*)$, let $D_\theta$ be the tempered distribution on $N$ with Fourier transform $\theta$.

**Theorem B.** For $\theta \in PB^\infty_N(n^*)$, $f \in \mathcal{P}(N)$, and $\xi \in n^*$,
\[
\pi_\xi(D_\theta \ast f) = \theta(\xi) \pi_\xi(f).
\]

As an application of these results, we consider the question of local solvability. Recall that a left invariant differential operator $L$ on $N$ is said to be locally solvable if there is an open set $U \subset N$ such that $C^\infty_c(U) \subset L(C^\infty_c(U))$.

Let $o(\xi)$ denote the $\text{Ad}^*$-orbit in $n^*$ that contains $\xi$, and having fixed a norm on $n^*$, set $|o(\xi)| = \inf\{||\xi'||: \xi' \in o(\xi)\}$. Suppose that $N$ contains a discrete, cocompact subgroup $\Gamma$. Then $L^2(\Gamma \backslash N)$ is a direct sum of subspaces $H_\xi$ such that the restriction to $H_\xi$ of right translation is a finite multiple of $\pi_\xi$. We denote by $(\Gamma \backslash N)_\xi^\circ$ the elements of $\tilde{N}$ appearing in this decomposition that are in general position.

**Theorem C.** Let $L$ be a left invariant differential operator on $N$. Suppose that for each $\pi_\xi \in (\Gamma \backslash N)_\xi^\circ$, $\pi_\xi(L)$ has a bounded right inverse $A_\xi$ on $H_\xi$, and that the norm of $A_\xi$ is bounded by a polynomial in $|o(\xi)|$. Then $L$ is locally solvable.

The proof of Theorem A requires the introduction of somewhat more general spaces. Let $H$ be a subspace of the center of $\mathfrak{n}$, and let $\lambda \in \mathfrak{h}^*$. We define the unitary character $\chi_\lambda$ on $H := \exp(H)$ by $\chi_\lambda(\exp X) = e^{2\pi i \langle \lambda, X \rangle}$, and denote by $\mathcal{P}(N/H, \chi_\lambda)$ the space of all smooth functions $f$ defined on $N$ such that $f(xy) = \chi_\lambda(y)f(x)$ for all $x \in N$, $y \in H$, and such that $f \circ \exp_{\mathfrak{h}} \in \mathcal{P}(H)$, where $\mathfrak{h}$ is a complement to $H$ in $\mathfrak{n}$. The topology of $\mathcal{P}(N/H, \chi_\lambda)$ is defined by requiring that the mapping $f \to f \circ \exp_{\mathfrak{h}}$ be a homeomorphism. Define $P_\lambda: \mathcal{P}(N) \to \mathcal{P}(N/H, \chi_\lambda)$ by
\[
P_\lambda f(\exp X) = \int_f f(\exp(X + Y))\chi_\lambda(-Y)\,dY.
\]
$P_\lambda$ is an open surjection and thus its adjoint $P^*_\lambda$ is an isomorphism of $\mathcal{S}^*(N/H, \chi_\lambda)$ into $\mathcal{S}^*(N)$.

Let $\mathcal{A}^\perp$ be the annihilator of $\mathcal{A}$ in $\mathfrak{a}^*$. For $\lambda \in \mathfrak{h}^*$ (identified as a subspace of $\mathfrak{a}^*$), there is a natural Schwartz space on $\mathcal{A}^\perp + \lambda$, $\mathcal{S}(\mathcal{A}^\perp + \lambda)$, given by composing elements of $\mathcal{S}(\mathcal{A}^\perp)$ with translation by $-\lambda$. Considering $\mathcal{S}(N/H, \chi_\lambda)$ and $\mathcal{S}(\mathcal{A}^\perp + \lambda)$ as subspaces of $\mathcal{S}^*(N)$ and $\mathcal{S}^*(\mathfrak{a}^*)$ respectively, the Fourier transform is defined on these spaces and one has that $f \mapsto \hat{f}$ is an isomorphism of $\mathcal{S}(N/H, \chi_\lambda)$ onto $\mathcal{S}(\mathcal{A}^\perp + \lambda)$ and of $\mathcal{S}(\mathcal{A}^\perp + \lambda)$ onto $\mathcal{S}(N/H, \chi_{-\lambda})$. Also one has that for $D \in \mathcal{S}^*(N/H, \chi_\lambda)$, $(P^*_\lambda D)^\wedge = R^*_{-\lambda} \tilde{D}$, where $R_\lambda : \mathcal{S}(\mathfrak{a}^*) \to \mathcal{S}(\mathcal{A}^\perp + \lambda)$ is restriction, and $\tilde{D}$ is the element in $\mathcal{S}^*(\mathcal{A}^\perp - \lambda)$ defined by $\langle \tilde{D}, f \rangle = \langle D, \hat{f} \rangle$. Thus $(P^*_\lambda D)^\wedge$ is supported on $\mathcal{A}^\perp + \lambda$ and has no normal derivatives.

For $f \in \mathcal{S}(N/H, \chi_\lambda)$ and $D \in \mathcal{S}^*(N/H, \chi_{-\lambda})$, the convolution $D \ast f$ is defined by setting $D \ast f(x) = \langle D, l_x(\hat{f}) \rangle$ for each $x \in N$. Suppose now that $D \in \mathcal{S}^*(N)$ and $f \in \mathcal{S}(N)$. One can use Abelian Fourier analysis to study the mapping defined on $\mathfrak{a}$, the center of $\mathfrak{a}$, by $Y \to D \ast f(\exp(X + Y))$. If this mapping is in $\mathcal{S}(\mathfrak{a})$, then

$$D \ast f(\exp X) = \int_{\mathfrak{a}^*} P_\lambda(D \ast f)(\exp X) \, d\lambda,$$

for appropriately normalized Lebesgue measure $d\lambda$. Furthermore, $P_\lambda(D \ast f) = D_\lambda \ast P_\lambda f$, where $D_\lambda$ is the element of $\mathcal{S}^*(N/H, \chi_{-\lambda})$ whose Fourier transform, $\tilde{D}_\lambda$, agrees with the restriction to $\mathcal{A}^\perp + \lambda$ of $\tilde{D}$. Thus, convolution between elements of $\mathcal{S}^*(N)$ and $\mathcal{S}(N)$ decomposes into convolutions between elements of $\mathcal{S}^*(N/H, \chi_{-\lambda})$ and $\mathcal{S}(N/H, \chi_\lambda)$ in such a way that smoothness and growth conditions on $\tilde{D}$, $D \in \mathcal{S}^*(N)$ are inherited by $\tilde{D}_\lambda$, $D_\lambda \in \mathcal{S}^*(N/H, \chi_{-\lambda})$. One then proceeds by induction on the dimension of $N/H$. Of course, this requires maintaining considerable control of the various seminorm estimates that appear in the decompositions.

The proof of Theorem B follows along the usual induction argument lines with the Plancherel Theorem being used to reduce the dimension.

For Theorem C, one constructs a $\theta$ on $\mathfrak{a}^*$ such that both $\theta$ and $1/\theta$ are in $PB_\mathcal{S}(\mathfrak{a}^*)$, and such that $\sum \|A(\xi)\|\theta(\xi) < \infty$, the sum being over $(\Gamma \backslash N)_0$. One then uses the fact that $(D_{1/\theta} \ast f) \ast (D_\theta \ast g) = f \ast g$ and the Dixmier and Mallivan [DM] factorization to complete the proof.

REMARKS. The fact that $D_\theta \in \mathcal{M}\mathcal{S}(N)$ was proved by R. Howe in [H], and indeed, the ideas presented there are the foundation of this work. Theorem B was proved for the case where $\theta$ is a polynomial by A. Kirillov in [K]. In [CG], L. Corwin and F. Greenleaf proved Theorem C with the additional assumption that all the representations in general position were induced from a common, normal subgroup. One-sided Schwartz multipliers have been studied by L. Corwin in [C].
REFERENCES


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