STRUCTURE THEORY AND REFLEXIVITY OF CONTRACTION OPERATORS

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1. Introduction. Let \( \mathcal{H} \) be a separable, infinite-dimensional, complex Hilbert space, and let \( \mathcal{L}(\mathcal{H}) \) denote the algebra of all bounded linear operators on \( \mathcal{H} \). The purpose of this note is to announce several new, and rather general, sufficient conditions that a contraction \( T \) in \( \mathcal{L}(\mathcal{H}) \) be reflexive, and, at the same time, to give various characterizations of the class of those contractions that possess an analytic invariant subspace (definition given below). Complete proofs and other results will appear in [7]. The principal new idea involved is a considerable improvement of the main construction of §3 of [9]. The new reflexivity theorems also depend on techniques from [9, 3, 1, and 4], and yield, in particular, the following improvement of the main result of [4].

**Theorem 1.1.** If \( T \) is a contraction in \( \mathcal{L}(\mathcal{H}) \) such that the spectrum \( \sigma(T) \) of \( T \) contains the unit circle \( \mathbb{T} \), then either \( T \) is reflexive or \( T \) has a nontrivial hyperinvariant subspace.

If \( T \in \mathcal{L}(\mathcal{H}) \) we denote by \( \mathcal{A}_T \) the dual algebra generated by \( T \) (i.e., \( \mathcal{A}_T \) is the smallest unital subalgebra of \( \mathcal{L}(\mathcal{H}) \) containing \( T \) that is closed in the weak* topology). It follows that \( \mathcal{A}_T \) is the dual space of \( Q_T = \mathcal{C}_1(\mathcal{H})/\perp \mathcal{A}_T \), where \( \perp \mathcal{A}_T \) is the preannihilator of \( \mathcal{A}_T \) in \( \mathcal{C}_1(\mathcal{H}) \), under the pairing

\[
(A, [L]) = \text{tr}(AL), \quad A \in \mathcal{A}_T, \; L \in \mathcal{C}_1(\mathcal{H}),
\]

where \( [L] \) denotes the element of the quotient space \( Q_T \) containing the trace-class operator \( L \). Thus, if \( x \) and \( y \) are vectors in \( \mathcal{H} \), then \( [x \otimes y] \) denotes the element of \( Q_T \) containing the rank-one operator \( x \otimes y \). The dual algebra \( \mathcal{A}_T \) is said to have property \((A_{1,\infty})\) if for any sequence \( \{[L_j]\}_{j=1}^{\infty} \) of elements from \( Q_T \) there exist vectors \( x \) and \( \{y_j\}_{j=1}^{\infty} \) in \( \mathcal{H} \) satisfying

\[
[L_j] = [x \otimes y_j], \quad j = 1, 2, \ldots.
\]

If, moreover, there exists \( \rho \geq 1 \) (independent of the family \( \{L_j\} \)) with the property that for every \( s > \rho \), the vectors \( \{x\} \) and \( \{y_j\} \) satisfying (1) can also be chosen to satisfy

\[
\|x\| \leq \left( s \sum_{k=1}^{\infty} \|L_k\| \right)^{1/2}, \quad \|y_j\| \leq (s\|L_j\|)^{1/2}, \quad j = 1, 2, \ldots,
\]

then we say that \( \mathcal{A}_T \) has property \((A_{1,\infty}(\rho))\).
Recall that if $T$ is an absolutely continuous contraction in $\mathcal{L}(H)$, and $H^\infty(T)$ is the usual Hardy algebra of functions on $T$, then the Sz.-Nagy-Foias functional calculus $\Phi_T: H^\infty(T) \to \mathcal{H}_T$ is a weak* continuous algebra homomorphism with range weak* dense in $\mathcal{H}_T$. The class $A = A(\mathcal{H})$ is defined to be the set of all those absolutely continuous contractions $T$ in $\mathcal{L}(H)$ for which $\Phi_T$ is an isometry; in other words, the set of such $T$ for which $\|f(T)\| = \|f\|_\infty$ for every $f$ in $H^\infty(T)$. Various sufficient conditions for an absolutely continuous contraction $T$ to belong to $A$ are known [2]. One such is that $\sigma(T) \cap D$ is dominating for $T$, where $D$ is the open unit disc in $C$. The class $A_{1,N_0}$ [resp. $A_{1,N_0}(\rho)$] is defined to consist of those $T$ in $A(\mathcal{H})$ for which $\mathcal{H}_T$ has property $(A_{1,N_0})$ [resp. $(A_{1,N_0}(\rho))$].

2. Analytic invariant subspaces. It turns out that another concept plays a central role in the derivation of our results—namely, the notion of an analytic invariant subspace (cf. [10, 3]). If $T$ is a contraction in $\mathcal{L}(H)$, $\mathcal{M} \in \text{Lat}(T)$, and there exists a nonzero conjugate analytic function $\epsilon: \lambda \to \epsilon_\lambda$ from $D$ into $C$, then $\mathcal{M}$ is said to be an analytic invariant subspace for $T$. If, in addition, $\forall \lambda \in D \epsilon_\lambda = \mathcal{M}$, then $\mathcal{M}$ is said to be a full analytic invariant subspace for $T$.

If $T \in \mathcal{L}(H)$, we write $\sigma_p(T)$, $\sigma_r(T)$, and $\sigma_0(T)$ for the point spectrum, right spectrum and essential (Calkin) spectrum of $T$ respectively. Moreover, following [8], we write $\mathcal{F}_p(T)$ for the set of all $\lambda$ in $C$ for which $T - \lambda$ is a Fredholm operator with (strictly) positive index. Recall also that a subspace $\mathcal{H}$ of $\mathcal{H}$ is said to be semi-invariant for $T$ if $\mathcal{H} = \mathcal{M} \ominus \mathcal{N}$, where $\mathcal{M}, \mathcal{N} \in \text{Lat}(T)$ and $\mathcal{M} \supset \mathcal{N}$; we denote the set of all semi-invariant subspaces for $T$ by $\mathcal{F}(T)$. (Of course, $\mathcal{H}$ itself and all elements of $\text{Lat}(T)$ belong to $\mathcal{F}(T)$.) As usual, if $\mathcal{H} \in \mathcal{F}(T)$, we write $T_\mathcal{H}$ for the compression of $T$ to $\mathcal{H}$.

**Theorem 2.1.** If $T$ is an absolutely continuous contraction in $\mathcal{L}(H)$, the following statements are equivalent:

(a) $T$ has an analytic invariant subspace.

(b) $T$ has a full analytic invariant subspace.

(c) $T \in A_{1,N_0}$.

(d) $T \in A_{1,N_0}(\rho)$ for some $\rho \geq 1$.

(e) There exists $\mathcal{H} \in \mathcal{F}(T)$ such that $\sigma_p(T_\mathcal{H}) = D$.

(f) There exists $\mathcal{H} \in \mathcal{F}(T)$ such that $T_\mathcal{H}$ is a Fredholm operator with (strictly) positive index.

Once and for all, we denote the set of all semi-invariant subspaces for $T$ by $\mathcal{S}_T(T)$.

The deeper ones depend on additional, more technical, characterizations of the class $A_{1,N_0}$ in terms of certain properties $E_{\theta,\gamma}$ and $F_{\theta,\gamma}$ which appear in [9 and 7], as well as on techniques and results from [8, 4 and 5].
3. Results on reflexivity. Recall that an operator $T$ in $\mathcal{L}(\mathcal{H})$ is said to be reflexive if every operator $S$ in $\mathcal{L}(\mathcal{H})$ such that $\text{Lat}(S) \supset \text{Lat}(T)$ belongs to $\mathcal{H}_T$, the closure of $\mathcal{H}_T$ in the weak operator topology. If $T$ is a contraction, we denote by $T_a$ the direct summand of $T$ that is the absolutely continuous part of $T$ (i.e., $T_a$ is the direct sum of the completely nonunitary part of $T$ and the absolutely continuous part of the unitary part of $T$).

**Theorem 3.1.** Each of the following is a sufficient condition that an arbitrary contraction $T$ in $\mathcal{L}(\mathcal{H})$ be reflexive:

(A) $T$ (or $T^*$) satisfies any one of the conditions (a)–(f) of Theorem 2.1.
(B) $T_a$ (or $T_a^*$) satisfies (c) or (d) of Theorem 2.1.
(C) $T_a \in (C_0, \cup C_0) \cap A$.
(D) $T_a \in (C_1, \cup C_1) \cap A$.
(E) $T$ is hyponormal and $T_a \in A$.

Theorem 1.1 follows from Theorem 3.1(C) via the fact that any contraction $T$ with $\sigma(T) \supset T$ not in the class $(C_0, \cup C_0) \cap A$ has nontrivial hyperinvariant subspaces (cf. [2, Theorem 4.3]), and on the basis of Theorem 3.1 we make the following conjectures.

**Conjecture 3.2** [6]. Every $T$ in $A$ is reflexive.

**Conjecture 3.3.** Every hyponormal operator is reflexive.

**References**


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