AN INVARIANT APPROACH TO THE THEORY
OF LOGARITHMIC KODAIRA DIMENSION
OF ALGEBRAIC VARIETIES

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Let $V$ be an algebraic variety defined over a field $k$. If $K$ is the rational function field of $V$, then $V$ is called a model of $K/k$, and the local ring of a point of $V$ is a locality of $V$. Let $L(K/k)$ be the set of discrete valuation rings of $K/k$. Define

$$\tilde{L}(V) = \{ R \in L(K/k) | R \text{ dominates a locality of } V \},$$

$$L(V) = \{ R \in L(K/k) | R \text{ is a locality of the normalization } \bar{V} \text{ of } V \}.$$  

If $V'$ is another model of $K/k$ and $\tilde{L}(V) = \tilde{L}(V')$, then we say that $V$ and $V'$ are proper birationally equivalent. The logarithmic Kodaira dimension $\kappa(V)$ of $V$ introduced by Iitaka (see [1]) is one of the most important proper birational invariants of $V$. Iitaka's treatment requires Hironaka's theory of resolution of singularities, and therefore at present does not apply to the cases of positive characteristics. In this note we shall describe a simple invariant approach to the theory of logarithmic Kodaira dimension of algebraic varieties defined over an arbitrary base field.

I would like to thank Professor T. Matsusaka for his encouragement throughout the stages of development of these results.

1. A divisor of $K/k$ is by definition a map $w: L(K/k) \to \mathbb{Z} \cup \{+\infty\}$ such that $w^{-1}(\mathbb{Z} - \{0\}) \cap L(V)$ is a finite set for one (therefore for any) model $V$ of $K/k$; $w$ is called absolute (denoted by $w \geq 0$) if $w(R) \geq 0$ for all $R \in L(K/k)$. For any $u \in K$ we define the principal divisor $(u)_{K/k}$ of $K/k$ by $(u)_{K/k}(R) = v_R(u)$ for all $R \in L(K/k)$, where $v_R$ is the normalized discrete valuation of $K/k$ determined by $R \in L(K/k)$. The divisors of $K/k$ form an abelian semigroup under pointwise addition. Two divisors $w$ and $w'$ of $K/k$ are linearly equivalent (notation: $w \sim w'$) if $w = w' + (u)_{K/k}$ for some $u \in K$.

Let $V$ be a model of $K/k$. We define two divisors $S_V$ and $T_V$ of $L(K/k)$ by the following rules:

$$\begin{cases}
S_V(R) = 0 & \text{for } R \in \tilde{L}(V), \\
S_V(R) = +1 & \text{for } R \notin \tilde{L}(V)
\end{cases} \quad \begin{cases}
T_V(R) = 0 & \text{for } R \in L(V), \\
T_V(R) = +\infty & \text{for } R \notin L(V).
\end{cases}$$

If $w$ is a divisor of $K/k$ we define

$$\bar{w}_V = w + S_V, \quad w_V = w + T_V.$$
Let $\Gamma(K/k)$ be the set of pairs $(R, R')$ of regular localities of $K/k$ such that $R$ dominates $R' \subset R$ and $\text{krulldim } R = 1$. Any absolute divisor $w$ of $K/k$ determines a map $r_w : \Gamma(K/k) \to \mathbb{Z}$ by

$$r_w(R, R') = w(R) - v_R(u)$$

where $u$ is a local function of $w$ at $R'$ (i.e. $v_{R''}(u) = w(R'')$ for any $R'' \in L(\text{spec}(R'))$). We call $r_w$ the ramification index of $K/k$ determined by $w$.

An absolute divisor $w$ of $K/k$ is called proper birationally invariant if for any $(R, R') \in \Gamma(K/k)$ we have $r_w(R, R') \geq \text{krulldim } R' - 1$.

Given any dominating pair $(R, R')$ of regular local rings such that

$$\text{krulldim } R = 1$$

and the quotient field of $R$ is a finite separable extension of the quotient field of $R'$, we introduce two invariants of $(R, R')$: 

$$r(R, R') = v_R(d(R/R'))$$

and

$$e(R, R') = \max \{v_R(u_1, \ldots, u_r) \mid (u_1, \ldots, u_r) \text{ is a minimal basis of the maximal ideal of } R'\}.$$ 

The integers $r(R, R')$ and $e(R, R')$ are called the ramification index and the reduced ramification index of $(R, R')$ respectively.

In case that $\text{krulldim } R' = 1$ we have $r(R, R') \geq e(R, R') - 1$ by the main theorem of ramification theory of algebraic number theory due to Dedekind. In [3] we proved that this is true in general, i.e.,

$$r(R, R') \geq e(R, R') - 1 \geq \text{krulldim } R' - 1.$$ 

(see [4] for an application of this formula).

Now back to our birational situation. We have the following theorem.

**THEOREM 1.1.** If $w$ is a proper birationally invariant divisor of $K/k$, then $r_w(R, R') \geq r(R, R') \geq e(R, R') - 1$ for any $(R, R') \in \Gamma(K/k)$.

2. We shall fix a polynomial ring $A = \bigoplus_{i=0}^{\infty} A_i = K[X]$ in one variable $X$ over $K$. For any divisor $w$ of $K/k$ and $m = uX^i \in A_i$ we let $w(m) = (u)_{K/k} + iw$ (we assume $0 \cdot (+\infty) = +\infty$); put $C_i(w) = \{m \in A_i \mid w(m) \geq 0\}$, $C(w) = \bigoplus_{i=0}^{\infty} C_i(w)$, $Z(w) = QC(w) \cap K$ where $QC(w)$ is the quotient field of $C(w)$, and $\kappa(w) = \text{trans. deg } C(w)/k - 1$. Define $\overline{Z}(w) = \bigcap Z(w_V)$ and $\overline{\kappa} = \min \kappa(w_V)$, where $V$ runs through the set of models of $K/k$. One can prove the following theorem easily (cf. [2]).

**THEOREM 2.1.** If $w$ and $w'$ are linearly equivalent divisors of $K/k$ then

$C(w) \cong C(w')$; $C(w)$ is an integrally closed $k$-graded algebra; $Z(w)$ and $\overline{Z}(w)$ are algebraically closed in $K$; if $w$ is absolute, then $\dim_k C_i < +\infty$.

Suppose $X$ is a model of $K/k$ and $D$ a reduced divisor of the normalization $\overline{X}$ of $X$. If $\mathcal{L}(V) = \mathcal{L}(\overline{X} - \sup D)$, then the pair $(X, D)$ is called a model of $V$; if $D = 0$ we also say that $X$ is a model of $V$. A model $(X, D)$ of $V$ is regular if $X$ is nonsingular, $D$ is a sum of nonsingular subvarieties and $\sup D$ has only normal crossings.
With the help of Theorem 1.1 we can prove the following

**Theorem 2.2.** If \( w \) is a proper birationally invariant absolute divisor of \( K/k \) and \((X, D)\) a regular complete model of a model \( V \) of \( K/k \), then

\[
C(\tilde{w}_{\nu}) = C((\tilde{w}_{\nu})_x) = \bigoplus_{i=0}^{\infty} H^0(X, \mathcal{O}_X(i(w(X) + D)));
\]

where \( w(X) = w|_{L(X)} \) is the Weil divisor of \( X \) induced by \( w \), \( Z(\tilde{w}_{\nu}) = Z((\tilde{w}_{\nu})_x) \) and \( \kappa(\tilde{w}_{\nu}) = \kappa((\tilde{w}_{\nu})_x) \).

**Proof.** It suffices to prove that \( C(\tilde{w}_{\nu}) = C((\tilde{w}_{\nu})_x) \) because then all the other assertions follow by definitions. Since \( \tilde{w}_{\nu} \leq (\tilde{w}_{\nu})_x \), \( C(\tilde{w}_{\nu}) \subseteq C((\tilde{w}_{\nu})_x) \), so we only need to prove \( C((\tilde{w}_{\nu})_x) \subseteq C(\tilde{w}_{\nu}) \).

To simplify notations we write \( w' \) for \( w_{\nu} \) and \( w'' \) for \( (\tilde{w}_{\nu})_x \). For any \( P \in L(K/k) \) let \( R' \) be the local ring of the center \( P \) of \( v_R \) on \( X \). Then \( P \in \text{sup } D \) if and only if \( R' \notin \tilde{L}(V) \). Let \((u_1, \ldots, u_r)\) be a minimal basis of the maximal ideal of \( R' \) such that, if \( P \in \text{sup } D \), \((u_1, \ldots, u_t)\) is a set of local equations of the divisor \( D \) at \( P \) for some \( 1 \leq t \leq r \). Write \( a = u_1 \cdots u_r \) and \( b = u_1 \cdots u_t \) (if \( R' \in \tilde{L}(V) \) we let \( b = 1 \)). Let \( u' \) be a local function of \( w'_{\nu} \).

Now suppose \( m = uX^t \in C_i(w'_{\nu}) \). We have to prove that \( m \in C_i(w') \), i.e., \( w'(m)(R) \geq 0 \) for any \( R \in L(K/k) \). For any \( R'' \in L(\text{spec}(R')) \) we have \( w''(m)(R'') = v_{R''}(u(u'b)^{t}) \geq 0 \), which implies that \( u(u'b)^{t} \in \bigcap R'' = R' \subseteq R \). It follows that \( v_R(u) + i(v_R(u') + v_R(b)) \geq 0 \). But \( w(R) - v_R(u') \geq v_R(a) - 1 \) by Theorem 1.1. Hence

\[
(*) \quad v_R(u) + i(w(R) + 1 - v_R(a) + v_R(b)) \geq 0.
\]

Now recall the definition of \( w'(m)(R) \):

\[
\begin{align*}
w'(m)(R) &= v_R(u) + i(w(R) + 1) & \text{for } R \notin \tilde{L}(V), \\
\end{align*}
\]

\[
\begin{align*}
\text{if } R \in \tilde{L}(V) \text{ then } w'(m)(R) &\geq 0 \text{ because in } (*) -v_R(a) + v_R(b) \leq 0. \text{ If } R \in \tilde{L}(V) \text{ then } w'(m)(R) \geq 0 \text{ because in } (*) v_R(b) = v_R(1) = 0 \text{ and } 1 - v_R(a) \leq 0. \text{ Thus we have proved that } w'(m)(R) \geq 0 \text{ for any } R \in L(K/k). \text{ This finishes the proof.}
\end{align*}
\]

3. Let \( k' \) be a perfect subfield of \( k \) (e.g., the prime field of \( k \)) and \( D(K/k') \) the differential module of \( K \) over \( k' \). A subset \( B \) of \( K \) is called a \( k' \)-differential basis of \( K/k \). A subset \( B \) of \( K \) is called a \( k' \)-differential basis of \( K/k \) if \( dB \) is a \( K \)-linear basis for \( D(K/k') \) and \( B - B \cap k \) is a finite set. If \( R \in L(K/k) \) we proved in [2] that \( RdR \) is an \( R \)-free module and there exists a \( k' \)-differential basis \( B_R \) of \( K/k \) such that \( dB_R \) is an \( R \)-free basis for \( RdR; B_R \) is called a set of \( k' \)-uniformizing coordinates of \( R \).

For any two \( k' \)-differential bases \( B, B' \) of \( K/k \) one can define an element \( J(B, B') \in K \), uniquely determined by \( (B, B') \) up to a factor in the algebraic closure \( \bar{k} \) of \( k \) in \( K \), such that, if \( B, B' \) are two sets of \( k' \)-uniformizing coordinates for some \( R \in L(K/k) \), then \( J(B, B') \) is an invertible element of \( R \).

For any \( k' \)-differential basis \( B \) of \( K/k \) we define the divisor \((B)\) of \( K/k \) by \( (B)(R) = v_R(J(B, B_R)) \), where \( B_R \) is a set of \( k' \)-uniformizing coordinates of \( R \).
If $k''$ is another perfect subfield of $k$ and $B''$ a $k''$-differential basis of $K/k$, then one can show that $(B)$ and $(B'')$ are linearly equivalent. Any divisor of $K/k$ which is linearly equivalent to $(B)$ is called a canonical divisor of $K/k$. We summarize the main properties of the canonical divisors of $K/k$ in the following theorem.

**Theorem 3.1.** (1) If $F$ is a subfield of $K$ containing $k$, then $(B)|_{L(K/F)}$ is a canonical divisor of $K/F$; (2) $(B)$ is proper birationally invariant.

4. Let $V$ be a model of $K/k$ and $w$ a canonical divisor of $K/k$. It is easy to see that $C(w), Z(w), Z(w), \kappa(w), \kappa(w)$ are proper birational invariants of $V$, denoted by $C(V), Z(V), \kappa(V), \kappa(V)$ respectively. If $V$ is complete, then $w = w$, therefore $C(V), Z(V), \ldots$ are all birational invariants of $V$, denoted by $C(K/k), Z(K/k), \ldots$ respectively.

**Definition 4.1.** $C(V)$ and $K(V)$ are called the (logarithmic) Kodaira dimension of $V$ and $K/k$ respectively; $\kappa(V)$ and $\kappa(K/k)$ are the virtual (logarithmic) Kodaira dimension of $V$ and $K$ respectively.

Since canonical divisors of $K/k$ are proper birationally invariant, we see from Theorem 2.2 that our definition of $\kappa(V)$ is equivalent to that of Iitaka's whenever the latter is applicable (notice that $\kappa(V)$ is usually denoted by $\bar{\kappa}(V)$).

Let $F$ be a subfield of $K$ containing $k$. If $U = \text{spec} B$ is an affine open subset of $V$, then $U' = \text{spec} F(B)$ is an affine model of $K/F$, here $F(B)$ is the affine ring of $K/F$ generated by the affine ring $B$ over $F$. The collection of all such $U'$ defines a model $V_{K/F}$ of $K/F$. Applying Theorem 3.1(1) we can prove the following

**Theorem 4.2.** (1) If $\bar{\kappa}(V) \geq 0$ then $\bar{\kappa}(V_{K/Z(V)}) = 0$; (2) If $\bar{\kappa}(V_{K/F}) = 0$ then $F \supseteq Z(V)$; (3) $\kappa(V) \leq \kappa(V_{K/F}) + \dim F/k$ and $\bar{\kappa}(V) \leq \bar{\kappa}(V_{K/F}) + \dim F/k$.

**Theorem 4.3.** Any $K/k$ can be uniquely factored into a series of extensions: $k \subseteq \bar{k} = F_0 \subsetneq F_1 \subsetneq F_2 \cdots \subsetneq F_{r-1} \subsetneq F_r = K$, $0 \leq r \leq \dim K/k$ such that (1) every $F_i$ is algebraically closed in $K$; (2) $\bar{\kappa}(F_1/F_0) \leq 0$ or $\bar{\kappa}(F_1/F_0) = \dim F_1/F_0$; (3) $\bar{\kappa}(F_i/F_i-1) = 0$ for $1 < i \leq r$; (4) $\bar{\kappa}(F_i/F_0) = \dim F_i/F_0$ for $1 < i \leq r$.

When $\text{ch } k = p > 0$ for geometric reasons it is important to know whether $K/Z(K/k)$ is a regular extension in the case that $0 < \kappa(K/k) < \dim K/k$. In this respect we have the following.

**Theorem 4.4.** Suppose $p \neq 2, 3$, and $\bar{\kappa}(K/k) = \dim K/k - 1$. Then $K/Z(K/k)$ is a regular extension.

**Proof.** We have $\dim K/Z(K/k) = 1$ and $\kappa(K/Z(K/k)) = \kappa(K/Z(K/k)) = 0$ by Theorem 4.3. Thus the genus of $K/Z(K/k)$ is 1. According to [5], the genus $g$ of an inseparable algebraic function field of one variable of characteristic $p > 0$ satisfies the relation $2g \geq p(p - 3) + 2$. In our case $g = 1$ and $p \neq 2, 3$. It is immediate that $K/Z(K/k)$ must be a separably generated extension, hence a regular extension.
COROLLARY 4.5. Suppose $k$ is perfect and $p \neq 2, 3$, and $\dim K/k \leq 3$. Then $K/\mathbb{Z}(K/k)$ is separably generated.

5. Let $V$ be a model of $K/k$ and $P(V) = \{f \in \text{Aut}_k K \mid \text{the map } f' : L(K/k) \to L(K/k) \text{ induced by } f \text{ maps } L(V) \text{ onto } L(V)\}$. If $V = \text{spec } B$ is a normal affine model of $K/k$ then $B = \bigcap_{R \in \mathcal{L}(V)} R$, hence $f \in P(V)$ if and only if $f(B) = B$; therefore $P(V) = \text{Aut}_k(B)$.

THEOREM 5.1. Assume $k$ is algebraically closed. (1) If $\kappa(K/k) = \dim K/k$, then $\text{Aut}_k K$ is a finite group. (2) If $\kappa(V) = \dim V$, then $P(V)$ is a finite group. (3) Suppose $V = \text{spec } B$ and $V$ has a regular complete model. If $\kappa(K/k) \geq 0$, then $P(V)$ is a finite group.

The assertion (3) follows directly from (2) since $\kappa(V) = \dim V$ under the assumption; (2) is due to Itaka when $\text{ch } k = 0$, which generalizes the classical result that, if $\dim K/k = 1$ and the genus of $K/k$ is 1, then the group of all automorphisms of $K/k$ that leaves a given place of $K/k$ fixed is finite. Finally (1) is well known when $K/k$ has a nonsingular complete model.

BIBLIOGRAPHY
